INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical and Risk Modelling - 2

Chapter: Unit-3

Category: Practice Question

1. CT6 APRIL 2010 Q10

Claims on a portfolio of insurance policies arrive as a Poisson process with annual rate λ. Individual claims are for a fixed amount of 100 and the insurer uses a premium loading of 15%. The insurer is considering entering a proportional reinsurance agreement with a reinsurer who uses a premium loading of 20%. The insurer will retain a proportion α of each risk.

(i) Write down and simplify the equation defining the adjustment coefficient R for the insurer.

(ii) By considering R as a function of α and differentiating show that:

$$\frac{(120\alpha - 5)dR}{d\alpha} + 120R = \left(100R + \frac{100\alpha dR}{d\alpha}\right) e^{100\alpha R}$$

(iii) Explain why setting $\frac{dR}{d\alpha} = 0$ and solving for α may give an optimal value for α.

(iv) Use the method suggested in part (iii) to find an optimal choice for α.

2. CT6 SEPT 2010 Q2

Claims on a portfolio of insurance policies follow a compound Poisson process with annual claim rate λ. Individual claim amounts are independent and follow an exponential distribution with mean μ. Premiums are received continuously and are set using a premium loading of θ. The insurer's initial surplus is U.

Derive an expression for the adjustment coefficient, R, for this portfolio in terms of μ and θ.

3. CT6 SEPT 2010 Q8

Claims on a portfolio of insurance policies arrive as a Poisson process with rate λ. The claim sizes are independent identically distributed random variables X_1, X_2, …with:

$$P(X_i = k) = p_k \text{ for } k = 1,2,\dots,M \text{ and } \sum_{k=1}^{M} p_k = 1.$$

The premium loading factor is θ.

(i) Show that the adjustment coefficient R satisfies:

$$\frac{1}{M}\log(1+\theta) < R < \frac{2\theta m_1}{m_2}$$

where $m_i = E(X_1^i)$ for $i = 1, 2$. [7]

[The inequality $e^{Rx} \le \frac{x}{M}e^{RM} + 1 - \frac{x}{M}$ for $0 \le x \le M$ may be used without proof.]

(ii) (a) Determine upper and lower bounds for R if θ = 0.3 and Xi is equally likely to be 2 or 3 (and cannot take any other values).

(b) Hence derive an upper bound on the probability of ruin when the initial surplus is U.

Answer: (ii) a) 0.08745 0.23077, b)e^-0.08745U

4. Subject CT6 April 2011 Q9

Claims on a portfolio of insurance policies arise as a Poisson process with parameter λ. Individual claim amounts are taken from a distribution X and we define $m_i = E(X^i)$ for i = 1, 2.... The insurance company calculates premium using a premium loading of θ.

(i) Define the adjustment coefficient R.

(ii) (a) Show that R can be approximated by $\frac{2\theta m_1}{m_2}$ by truncating the series expansion of $M_X(t)$

(b) Show that there is another approximation to R which is a solution of the equation $m_3 y^2 + 3m_2 y - 6\theta m_1 = 0$

Now suppose that X has an exponential distribution with mean 10 and that θ=0.3

(iii) Calculate the approximations to R in (ii)a and (ii)b and compare them to the value of R.

Answer: (iii) R = 0.03, R = 0.0230769

5. CT6 OCT 2011 Q4

Claims on a portfolio of insurance policies follow a Poisson process with parameter λ. The insurance company calculates premiums using a premium loading of θ and has an initial surplus of U.

(i) Define the surplus process U(t).

(ii) Define the probabilities ψ(U, t) and ψ(U).

(iii) Explain how ψ(U, t) and ψ(U) depend on λ.

6. CT6 OCT 2011 Q9

Claim events on a portfolio of insurance policies follow a Poisson process with parameter λ. Individual claim amounts follow a distribution X with density

$$f(x) = 0.01^2 x e^{-0.01x} \qquad x > 0.$$

The insurance company calculates premiums using a premium loading of 45%.

(i) Derive the moment generating function MX(t).

(ii) Determine the adjustment coefficient and hence derive an upper bound on the probability of ruin if the insurance company has initial surplus U.

(iii) Find the surplus required to ensure the probability of ruin is less than 1% using the upper bound in (ii).

Suppose instead that individual claims are for a fixed amount of 200.

(iv) Determine whether the adjustment coefficient is higher or lower than in (ii) and comment on your conclusion.

Answer: (ii) R = 0.00215578, e^-0.00215578U, (iii) U > 2136.20

7. CT6 April 2012 Q11

Claims on a portfolio of insurance policies arrive as a Poisson process with parameter 100. Individual claim amounts follow a normal distribution with mean 30 and variance 5^2 . The insurer calculates premiums using a premium loading of 20% and has an initial surplus of 100.

(i) Define carefully the ruin probabilities $\psi(100)$, $\psi(100,1)$ and $\psi_1(100,1)$.

(ii) Define the adjustment coefficient R.

(iii) Show that for this portfolio the value of R is 0.011 correct to 3 decimal places.

(iv) Calculate an upper bound for $\psi(100)$ and an estimate of $\psi_1(100,1)$.

(v) Comment on the results in (iv).

Answer: (iv) exp(-100 x 0.011) = 0.33287, $\psi_1(100,1) = 0.0107$

8. CT6 September 2012 Q10

Claims occur on a portfolio of insurance policies according to a Poisson process. Individual claim amounts are either 1 (with probability 0.7) or 8 (with probability 0.3). The insurance company uses a premium loading of 60% to calculate premiums and buys excess of loss reinsurance with a retention of M ($1<M<8$) from a reinsurer.
The reinsurer uses a premium loading of 120%.

(i) Calculate the smallest value of M that the insurance company should consider if it wishes to expect to make a profit on this portfolio.

(ii) Derive the adjustment coefficient equation for the insurance company.

(iii) Calculate the adjustment coefficient (correct to 2 decimal places) if M=4.

The same reinsurer also offers proportional reinsurance with the same premium loading such that the reinsurer pays a proportion α of each claim.

(iv) Show that the insurance company may either purchase excess of loss reinsurance with retention M or proportional reinsurance with $\alpha = \frac{3(8-M)}{31}$ for the same premium.

(v) Determine whether the adjustment coefficient with proportional reinsurance is higher or lower than that with excess of loss reinsurance when M=4.

(vi) Comment on the implications of part (v).

Answer: (i) M > 2.833, (iii) R = 0.13

10. CT6 April 2013 Q9

Claims on a portfolio of insurance policies arise as a Poisson process with rate λ. The mean claim amount is μ. The insurance company calculates premiums using a loading of θ and has an initial surplus of U.

(i) Explain how the parameters λ, μ, θ and U affect ψ (U), the probability of ultimate ruin.

Now suppose that λ=50, μ=200 and θ=30%. There are three models under consideration for the distribution of individual claim amounts:

A. fixed claims of 200
B. exponential with mean 200
C. gamma with mean 200 and variance 800

Let the corresponding adjustment coefficients be , R_A R_B and R_C .

(ii) Find the numerical value of R_B and show that R_B is less than both R_A and R_C

(iii) Use the fact that $\left(1+\frac{x}{n}\right)^n$ for large n to show that R_A and R_C are approximately equal.

11. CT6 September 2013 Q3

Andy is a famous weight lifter who will be competing at the Olympic Games. He has taken out special insurance which pays out if he is injured. If the injury is so serious that his career is ended the policy pays $1m and is terminated. If he is injured but recovers the insurance payment is $0.1m and the policy continues.

The insurance company's underwriters believe that the probability of an injury in any year is 0.2, and that the probability of more than one injury in a year can be ignored. If Andy is injured, there is a 75% chance that he will recover.

Annual premiums are paid in advance, and the insurance company pays claims at the end of the year. Assume that this is the only policy that the insurance company writes, and that it has an initial surplus of $0.1m.

(i) Define what is meant by ψ ($0.1m,1) and ψ ($0.1m).

(ii) Calculate the annual premium charged assuming the insurance company uses a premium loading of 30%.

(iii) Determine ψ ($0.1m,2).

Answer: (i) ψ ($0.1m,1) = $0.1m, ψ ($0.1m) = $0.1m, (ii) $0.0845m,
(iii) ψ ($0.1m,2) = 0.0975

12. CT6 April 2014 Q2

Ruth takes the bus to school every morning. The bus company's ticket machine is unreliable and the amount Ruth is charged every morning can be regarded as a random variable with mean 2 and non-zero standard deviation. The bus company does offer a "value ticket" which gives a 50% discount in return for a weekly payment of 5 in advance. There are 5 days in a week and Ruth walks home each day.

Ruth's mother is worried about Ruth not having enough money to pay for her ticket and is considering three approaches to paying for bus fares:

A. Give Ruth 10 at the start of each week.
B. Give Ruth 2 at the start of each day.
C. Buy the 50% discount card at the start of the week and then give Ruth 1 at the start of each day.

Determine the approach that will give the lowest probability of Ruth running out of money.

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Answer: Approach A

13. CT6 April 2014 Q5

A particular portfolio of insurance policies gives rise to aggregate claims which follow a Poisson process with parameter $\lambda = 25$. The distribution of individual claim amounts is as follows:

Claim	50	100	200
Probability	30%	50%	20%

The insurer initially has a surplus of 240. Premiums are paid annually in advance.

Calculate approximately the smallest premium loading such that the probability of ruin in the first year is less than 10%.

Answer: 0.1948

14. CT6 September 2014 Q8

Claims on a portfolio of insurance policies follow a Poisson process with rate λ. Individual claim amounts follow a distribution X with mean μ and variance $\sigma 2$. The insurance company charges premiums of c per policy per year.

(i) Write down the equation satisfied by the adjustment coefficient R.

(ii) Show that R can be approximated by

$$\hat{R} = \frac{2(c-\mu)}{\sigma^2 + \mu^2}$$

Now suppose that individual claims follow a distribution given by

Value	10	20	50	100
Probability	0.3	0.50	0.15	0.05

The insurance company uses a premium loading of 30%. It is considering the following reinsurance arrangements:

A. No reinsurance.

B. Proportional reinsurance where the insurer retains 70% of all claims with a reinsurer using a 20% premium loading.

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C. Excess of loss reinsurance with retention 70 with a reinsurer using a 40% premium loading.

(iii) Determine which arrangement gives the insurance company the lowest probability of ultimate ruin, using the approximation in part (ii)

(iv) Comment on your result in part (iii).

15. CT6 April 2015 Q10

Claims on a certain portfolio of insurance policies arise as a Poisson process with annual rate λ. Individual claim amounts are independent from claim to claim and follow an exponential distribution with mean μ. The insurance company has purchased excess of loss reinsurance with retention M from a reinsurer who calculates premiums using a premium loading of θ. Denote by X_i the amount paid by the reinsurer on the ith claim (so that $X_i = 0$ if the ith claim amount is below M).

(i) Explain why the claims arrival process for the reinsurer is also a Poisson process and specify its parameter.

(ii) Show that

$$M_{X_i}(t) = 1 + e^{-M/\mu} \times \frac{\mu t}{1-\mu t}.$$

(iii) (a) Determine E(Xi).
(b)Write down and simplify the equation for the reinsurer's adjustment coefficient.

(iv) Comment on your results to part (iii).

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PRACTICE QUESTION

16. CT6 April 2016 Q7

Claims on a portfolio of insurance policies arrive as a Poisson process with parameter λ, claim amounts having a Normal distribution with mean μ and variance σ^2, and there is a loading θ on premiums. The insurance company has an initial surplus of U.

(i) Explain carefully the meaning of $\psi(U)$, $\psi(U,t)$ and $\psi(U,1)$.

(ii) State four factors that affect the size of $\psi(U,t)$, for a given t.

(iii) Explain, for each factor, what happens to $\psi(U,t)$ when the factor increases.

Sarah, the insurance company's actuary, prefers to consider the probability of ruin in discrete rather than continuous time.

(iv) Explain the advantages and disadvantages of Sarah's approach.

17. CT6 September 2016 Q10

Claims on a portfolio of insurance policies arise as a Poisson process with parameter λ = 125. Individual claim amounts, X_i follow a gamma distribution with parameters α = 20 and β = 0.5.

The insurance company calculates premiums using a premium loading factor of 15% and has an initial surplus of 300.

(i) Define the adjustment coefficient R.

(ii) Show that for this portfolio the value of R is 0.00648 correct to three significant figures.

(iii) (a) Calculate an upper bound for $\Psi(300)$.

(b) Calculate an estimate of $\Psi_1(300,1)$, using a Normal approximation.

The parameter β now reduces to 0.4.

(iv) Explain what would happen to the estimate of $\Psi_1(300,1)$, without carrying out any further calculations.

(v) Propose two ways in which the insurance company could reduce $\Psi_1(300,1)$.

Answer: (iii) a) 0.143, b) 0.011

18. CT6 April 2017 Q10

Total annual claim amounts S on a portfolio of insurance policies come from two independent types of policies:

Type I, which have claim amounts uniformly distributed between 3,000 and 4,000.

Type II, which have claim amounts following an Exponential distribution with mean 3,600.

Claims occur according to a Poisson process, with mean 15 per annum for Type 1 claims and mean 25 per annum for Type 2 claims.

The insurance company uses a premium loading factor of 7% and checks for ruin at the end of each year.

(i) Calculate the mean and standard deviation of S.

(ii) Calculate the minimum initial surplus U_m required such that the probability of ruin at the end of the first year is less than 0.015, using a Normal approximation for the distribution of S.

Regulatory reforms mean the insurance company is trying to reduce this probability of ruin to less than 0.005. The insurance company is therefore purchasing proportional reinsurance from a reinsurer, who uses a premium loading factor of 17% in its premiums.

The insurance company retains a proportion α of each claim, and denotes by S_I the aggregate annual claims it retains net of reinsurance. The insurance company continues to hold initial surplus U_m.

(iii) Calculate the maximum proportion α_{max} that the insurance company can retain in order to keep the probability of ruin less than 0.005, using a Normal approximation for the distribution of S_I.

The insurance company is concerned that α_{max} is too low, reducing its profits, and intends to retain a higher proportion.

(iv) Suggest other ways in which the insurance company can reduce the probability of ruin.

Answer: (i) E(S) =142,500, SD(S) = 28,862, (ii) U > 52,658, (iii) α < 76.6%

19. CT6 September 2017 Q3

On 1 January 2014 an insurance company writes a policy for a European farmer. At the end of each year, the farmer's crop is assessed, and if it is less than 100 tonnes, it is deemed to have failed. If the crop fails for two years in a row the insurance policy pays out €1m and then is immediately terminated.

Premiums are €25k per month, paid in advance, and there are no expenses. This is the only policy the insurance company writes and it has initial surplus U > 0. Denote by Ψ(U,t) the probability of ruin by time t, measured in years; and Ψ(U) as the ultimate probability of ruin.

Explain whether the following statements are TRUE or FALSE:

(a) $\Psi(U,1) < \Psi\left(U,1\frac{1}{2}\right)$

(b) $\Psi\left(U,2\frac{1}{2}\right) = \Psi\left(U,3\frac{1}{3}\right)$

(c) $\Psi(U,3) < \Psi(U,4)$

(d) $\Psi(U,4) = \Psi(U)$

20. CT6 April 2018 Q8

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Claim events on a portfolio of insurance policies follow a Poisson process with parameter λ. Individual claim amounts, X, follow a Normal distribution with parameters $\mu = 500$ and $\sigma^2 = 200$.
The insurance company calculates premiums using a premium loading factor of 20%.

(i) Show that the adjustment coefficient, r = 0.000708 to three significant figures.

The insurance company's initial surplus is 5,000.
(ii) Calculate an upper bound on the probability of ruin, using Lundberg's inequality.

The insurance company actuary believes that claim amounts are better modelled using an exponential distribution. You may assume that the mean m is unchanged, and is now equal to the standard deviation.
(iii) Calculate the new upper bound of the probability of ruin.

(iv) Give a reason why claim amounts on insurance policies are not usually modelled using a Normal distribution, and suggest an alternative distribution, other than the exponential.

Answer: (ii) 0.029, (iii) 0.189

21. CT6 September 2018 Q7
Claims on a portfolio of insurance policies arise as a Poisson process with parameter λ. Individual claim amounts are taken from a distribution X and we define $m_i = E(X^i)$ for i = 1, 2, …. The insurance company calculates premiums using a premium loading of θ.
(i) Define the adjustment coefficient R.

(ii) Show that R can be approximated as $\frac{2\theta m_1}{m_2}$, by truncating the series expansion of $M_X(t)$.

Now suppose that X follows an exponential distribution with parameter γ.
(iii) Show that R = $\frac{\theta\gamma}{1+\theta}$.

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The insurance company uses a premium loading of 12%, and the mean claim amount is 200.
(iv) Calculate R, commenting on the difference with the approximation to R shown in part (ii).

The initial surplus is 5,000.
(v) Calculate an upper bound for the ultimate probability of ruin.

(vi) Suggest two methods by which the insurance company can reduce the probability of ruin.
Answer: (iv)0.000536 (v) 6.9%

23. CM2A September 2019 Q10
An insurer writes policies that insure policyholders against losing their mobile phone. Each policy pays £500 at the end of the year if the policyholder loses their phone during the year.

The insurer has carried out research on mobile phone losses in the general population, which showed that there is a 10% chance of an individual losing their phone in a one-year period. To allow for expected claims and a profit margin the insurer sets the annual premium at £125. Premiums are paid at the start of the policy year.

The insurer writes ten policies and has no assets at the time the policies are written. The insurer assumes that the policies are independent of each other. Discounting and expenses can be ignored.

(i) Calculate the probability that the insurer is still solvent at the end of the year.

After one year the insurer has received two claims.

(ii) State two factors relating to policyholder behaviour which may give rise to the insurer experiencing more claims than expected, explaining for each factor how it may arise in this case.

(iii) Explain how insurers typically reduce the risk posed by the factors identified in part (ii).

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At the end of year one, after paying any claims but before receiving any more premiums, the insurer has assets of £250. All ten policyholders remain with the insurer for another year and all still have a 10% chance per year of losing their phone.
The insurer decides that premiums for year two will be £x per policy, but with a 25% discount for any policyholders who did not claim in year one.

(iv) Calculate the base premium, £x, that the insurer should charge to achieve a probability of ruin in year two of no more than 5%.

Answer: (i) 0.9298, (iv) x = £156.25

24. CM2A September 2020 Q8

Consider an insurer with initial surplus U and surplus process U(t) at time t.

(i) (a) Define the surplus process for this insurer in terms of the initial surplus, premium income and the aggregate claims process, defining all notation used.
(b) State two limitations of this model.

(ii) (a) Define the probabilities of ruin in both finite and infinite time.
(b) State the relationship between these probabilities.

An insurance company has written 500 1-year car insurance policies. Each policy charges a monthly premium of £10 over the life of the policy. You can assume for your calculations that the premium is paid continuously.

The probability of a claim on each policy in any given month is 1%, and claims are accounted for on the last day of each month. Claim amounts follow a uniform distribution between £0 and £1,000. Claims are independent and each policy can have at most one claim per month.

(iii) Calculate the mean and standard deviation of the monthly aggregate claims.

Denote by $\psi(U, t)$ the probability of ruin before time t (measured in months), given initial surplus U.

(iv) Estimate $\psi(2{,}000, 1)$ by assuming that monthly claims are approximately Normally distributed.

(v) Comment on how the probability of ruin will develop for times $t > 1$.

Answer: (iii) E(S) = 2,500, Var(S) =1,654,167 , SD(S) = 1,286, (iv) 0 0.000234

25. CM2A April 2021 Q8

A general insurer specializing in catastrophes is trying to understand its future probability of ruin. It models future claims payments in millions of dollars over each yearly interval as Exponentially distributed with parameter $\lambda = 1$. The insurer assumes that the claims payments are independent each year.
The insurer holds an initial surplus of $3m and assumes no future premium income.

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(i) Calculate the probability that the insurer is solvent at the following times measured in years:
(a) t = 0
(b) t = 1.

(ii) State, without performing any further calculations, how your answers to part (i) would change if the insurer changed its net claims distribution to be Exponentially distributed with parameter $\lambda = 2$.

(iii) Discuss the appropriateness of the insurer's approach to assessing its future probability of ruin.

Answer: (i) a) = 1, b) 0.95021

26. CM2A September 2021 Q3
Claims on a portfolio of insurance policies arise as a Poisson process with rate λ. The insurance company calculates premiums using a loading of θ and has an initial surplus of U. The probability of ruin before time t is defined as $\Psi(U, t)$.

Suppose that $\theta = 0.1$ and the claim amounts follow an exponential distribution with mean $\mu = 0.5$.

Calculate the numerical value of R, the adjustment coefficient.

Answer: $R = 0.1818$

27. CM2A September 2023 Q8
Under certain assumptions, the adjustment coefficient r for an insurer's surplus process is a parameter that is the unique positive root of the following equations:

$$\lambda M_X(r) - \lambda - cr = 0$$

$$c = (1 + \theta)\lambda m_1$$

$M_X(r)$ is the moment generating function of the individual claim amount distribution.

(i) Write down the key assumption that is required to make these equations valid.

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(ii) Define the following terms from the equations above:
(a) λ
(b) c
(c) θ
(d) m_1

The insurer is looking at a product where claims take a value 1 with probability p, and 0 otherwise.

(iii) Show that the approximate relationship $r \approx 2\theta$ holds. You may assume r is small enough that terms of order r^3 and above can be discarded.

28. CM2A September 2023 Q11

A surplus process for an insurer is denoted as U(t) with initial surplus U.

(i) Define the surplus process for this insurer in terms of the initial surplus, premium income and aggregate claims process. All notations used should be defined.

An insurer writes a 1-year pet insurance policy. The policy covers 100 pets and will pay out $10 on each death of an insured pet during the policy term.

The pet mortality rate is assumed to be 2.5% per month with claims being independent and assumed to follow a Poisson process. Premiums are paid continuously at $29 per month.

The insurer has an initial surplus of $50.

(ii) Calculate the probability that the time to the first claim is longer than 1 month.

(iii) Calculate the expected value of U(t) at:

(a) the end of month 1.
(b) the end of the year.

(iv) Calculate the probability that the initial surplus and the insurer's premium income will be insufficient to cover the total claims in the first year.

The insurer increases the estimate of pet mortality to 3.25% per month.

(v) Comment on the effect that this increase in the mortality estimate will have on the:

(a) probability of ruin in the next year.

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(b) ultimate probability of ruin.

Answer: (ii) 8.2%, (iii) a) E(U(1)) = 54, E(U(12)) = 98, (iv) 4.6%