INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical & Risk Modelling - 3

Chapter:

Category: Assignment 1 Questions

1. With the increasing number of G3M2 virus cases in the country, two vaccine manufacturers in the country have quickly developed two different single-vial vaccines. These vaccine manufacturers have their own distribution channel, which are well connected through-out the country. These vaccines need to be stored in a temperature-controlled case and an agency has been entrusted with the responsibility of administering the vaccines to the interested public. Considering that the vaccines have been developed within a very short time, there are very few takers for the vaccine worrying the potential side-effects. Due to limited availability, an agency of a particular vaccine manufacturer is permitted to store only four vaccines.

The number of vials administered during a day by an agency is a random variable with the following discrete distribution:

No. of vials potentially administered in a day	Probability
0	50%
1	25%
2	25%

The probability of administration of greater than one vial in a day remains 50%. If the agency has no vaccines in stock at the end of a day, the agency contacts its supplier to order four more vaccines. The vaccines are delivered the following morning, before the agency opens. The vaccine supplier makes a charge of C for the delivery.

i) Write down the transition matrix for the number of vaccines in stock when the agency opens in a morning, given the number of vaccines when the agency opened the previous day.

ii) Calculate the stationary distribution for the number of vaccines in stock when the agency opens, using your transition matrix in part (i).

iii) Calculate the expected long-term average number of restocking orders placed by the agency per day.

iv) Calculate the expected long-term number of orders lost per day.

The agency is unhappy about losing these sales as there is a profit of P on each sale. It therefore considers changing its restocking approach to place an order before it has run out of vaccines. The charge for the delivery remains at C irrespective of how many vaccines are delivered.

v) Evaluate the expected number of restocking orders, and number of lost sales per trading day, if the agency decides to restock if there are fewer than two vaccines remaining in stock at the end of the day.

vi) Explain why restocking when two or more vaccines remain in stock cannot optimize the agency's profits.

The agency wishes to maximize the profit it makes on the vaccines.

vii) Derive a condition in terms of C and P under which the agency should change from only restocking where there are no vaccines in stock, to restocking when there are fewer than two vaccines in stock.

2. An annual subscription of a cloud-storage company follows a Markov chain with following states:

- State A: subscription continued
- State B: subscription cancelled by customer
- State C: subscription not renewed by the company

At t = 0, the number of subscribed customers = 500

Transition between states occur at end of year

T_{xy} indicates the transition matrix from x^{th} year to the y^{th} year.

T_{01} =

0.86	0.10	0.04
0	0	1
0	0	1

T_{12} =

0.84	0.11	0.05
0	0	1
0	0	1

Calculate the number of expected subscriptions cancelled by customers in the next 2 years.

3. The most popular T20 cricket league Hindustan Premier League (HPL) has seen two new entrants LSG and GT reaching the finals for the year 2022. It was a very closely fought contest, with each team scoring exactly 221 runs in their allotted 20 overs.

Now the winner for the tournament has to be decided through a modified version of super over, in order to identify the winner quickly. In this modified version of super over, the players of each team will be given 6 balls initially, in turns, to hit the wickets by bowling.

Bowler of each team is given chances in alternate fashion, one after the other. The team which hits the wickets will be given one point and in case it misses the wickets, a point will be awarded to the opposing team. At the end of allotted 6 balls, the team which has maximum points (either owing to hitting the wickets in its chances or due to other team missing the wickets in their chances), is judged the winner of HPL.

In case there is a tie, each team is asked to bowl additional balls and the team which has a clear lead of 2 points over the other, is judged the winner.

At the end of 20 balls, both LSG and GT stand tied at 10-10 each. If LSG hits the wicket in the next ball, then the score will be 11-10 in its favour and if GT misses the wicket in the following ball, then LSG would win the tournament.

In case LSG misses the wicket in the next ball (Score 10-11 in GT's favour) and GT hits the wicket the following ball after that, then GT would win the tournament. In case LSG wins the next point and GT wins the point after that, then the match is back tied. The probability of LSG winning a point is 60%.

The probability of GT winning a point, therefore, is 40%. You are asked to model the tied position in this case as a Markov chain.

i) Clearly specify the five different states of this Markov chain.

ii) Create a transition matrix based on the states obtained in (i) above.

iii) Calculate the number of points that must be played before there is more than 90% chance of the match being completed.

iv) Calculate the probability that LSG will eventually win the match and hence, the tournament.

v) Compare the result in (iv) with the probability of LSG winning a point and comment.

4. An insurance company employs 300 agents grouped into four grades labelled A, B, C and D. Agents move between the grades according to whether they meet their weekly business target. Agents employed at the start of any week remain employed throughout that week. At the end of each week, agents are considered for promotion to the next grades, or they leave employment. If an agent leaves the company, he is instantly replaced by a new one in grade A.

• Agents in grade A at the beginning of a week get promoted to grade B with probability 0.04, leave the company with probability 0.03 or continue in the same grade at the beginning of the next week.

• Agents in grade B at the beginning of a week get promoted to grade C with probability 0.03, leave the company with probability 0.06 or continue in the same grade at the beginning of the next week.

• Agents in grade C at the beginning of a week get promoted to grade D with probability 0.005, leave the company with probability 0.01 or continue in the same grade at the beginning of the next week.

• Agents in the grade D at the beginning of a week leave the company with the probability of 0.02 or continue in the same grade at the beginning of the next week.

i) Define Stationary probability distribution and its role in different Markov chains.

ii) Which of the following is Agents' movements transition probability matrix P (in terms of the grade they are in, and when they leave the company) of a discrete time Markov chain.

A. $$\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.96 & 0.04 & 0 & 0 \\ 0.06 & 0.97 & 0.03 & 0 \\ 0.01 & 0 & 0.995 & 0.005 \\ 0.02 & 0 & 0 & 0.98 \end{bmatrix} \end{array}$$

B. $$\begin{array}{c} \begin{array}{ccccc} L & A & B & C & D \end{array} \\ \begin{bmatrix} 0.03 & 0.96 & 0.04 & 0 & 0 \\ 0.06 & 0.97 & 0.03 & 0 & 0 \\ 0.01 & 0 & 0.985 & 0.005 & 0 \\ 0.02 & 0 & 0 & 0 & 0.98 \end{bmatrix} \end{array}$$

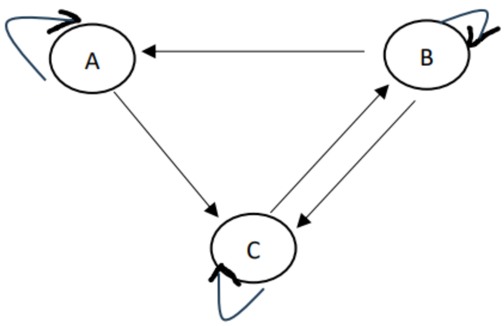
C. $$\begin{array}{c} \begin{array}{ccccc} L & A & B & C & D \end{array} \\ \begin{bmatrix} 0.03 & 0.93 & 0.04 & 0 & 0 \\ 0.06 & 0.03 & 0.97 & 0 & 0 \\ 0.01 & 0 & 0 & 0.995 & 0.005 \\ 0.02 & 0 & 0 & 0 & 0.98 \end{bmatrix} \end{array}$$

D. $$\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.96 & 0.04 & 0 & 0 \\ 0.06 & 0.91 & 0.03 & 0 \\ 0.01 & 0 & 0.985 & 0.005 \\ 0.02 & 0 & 0 & 0.98 \end{bmatrix} \end{array}$$

iii) Suppose the company has 300 agents at the beginning of week 1 distributed as 150 in grade A, 75 in grade B, 45 in grade C and 30 in grade D. Calculate the expected number of agents in each grade at the beginning of week 3. Explain the key steps in the solution.

iv) Calculate the expected number of agents in each grade in steady state.

5. A three state process with state space {A, B, C} is believed to follow a Markov chain with the following possible transitions:



An instrument was used to monitor this process, but it was set up incorrectly and only recorded the state occupied after every two time periods. From these observations the following two-step transition probabilities have been estimated:

$P^2[AA] = 0.5625$
$P^2[AB] = 0.125$
$P^2[BA] = 0.475$
$P^2[CC] = 0.4$

Calculate the one-step transition matrix consistent with these estimates.

6. An investor analyst assesses the rating of insurance and reinsurance companies every month. For the purpose of analysis, the ratings are grouped and classified in the order of merit as Marginal (rating symbols below B), Fair (rating symbols B and B-), Good (rating symbols B+ and B++), Excellent (rating symbols A and A-), Superior (rating symbols A+ and A++).

According to the history, the rating of the bonds evolves as a Markov chain with transition probability matrix given below for some parameter 'α' :

$$
\begin{array}{c|ccccc}
 & \mathbf{M} & \mathbf{F} & \mathbf{G} & \mathbf{E} & \mathbf{S} \\
\hline
\boldsymbol{M} & \alpha & \alpha & 1-2\alpha & 0 & 0 \\
\boldsymbol{F} & \alpha & 1-3\alpha-2\alpha^2 & 2\alpha & 2\alpha^2 & 0 \\
\boldsymbol{G} & 0 & 2\alpha^2 & 1-3\alpha-2\alpha^2 & 3\alpha & 0 \\
\boldsymbol{E} & 0 & 0 & 2\alpha^2 & 1-3\alpha-2\alpha^2 & 3\alpha \\
\boldsymbol{S} & 0 & 0 & 2\alpha^2 & 3\alpha & 1-3\alpha-2\alpha^2
\end{array}
$$

Note: The symbols M, F, G, E and S in the above matrix represent the rating groups Marginal, Fair, Good, Excellent and Superior respectively.

i) Draw the transition graph of the chain.

ii) Determine the range of α for which the matrix is a valid transition matrix.

iii) Explain whether the chain is irreducible and/or aperiodic.

iv) Using the above transition probability matrix, calculate the long-run probability that an insurance company is at rating status Superior (S). Assume α = 0.1.

7. i) In a city during rainy season the days were observed to be raining (R) or not raining (S) on any particular day. The data was recorded for a month of Jun2020 as below:

Week 1: RSRRSSS
Week 2: SRRSRSS
Week 3: RSRSRRS
Week 4: SSRRSSR
Week 5: RR

It was assumed that the rain prediction is dependent only on the previous day and hence decided to fit Markov chain.

a) Calculate transition probabilities for the Markov chain.

b) Determine the probability that it will rain on 3rd July 2020.

ii) A life insurance agent sells on average three life insurance policies per week. Use Poisson's law to calculate the probability that in a given week he will sell:

a) Some policies.

b) 2 or more policies but less than 5 policies.

c) Assuming that there are 5 working days per week, what is the probability that in a given day he will sell one policy?

8. A participating policy by an insurer 'ABC' pays reversionary bonus yearly per 1000 sum assured as follows:

State	Yearly bonus/1000
Business Flourish (BF)	9
Business De-grown (BD)	5
Business Stagnant (BS)	0
Business Expansion (BE)	3

The transition of the business states is as per the below matrix:

$$P = \begin{array}{c|cccc} & BF & BD & BS & BE \\ \hline BF & 0.2 & 0.8 & 0 & 0 \\ BD & 0 & 0.7 & 0.3 & 0 \\ BS & 0 & 0 & 0.4 & 0.6 \\ BE & 0.4 & 0 & 0 & 0.6 \end{array}$$

Annual interest rate = 0% and transition happens after bonus is paid.

(i) Determine the period of each of the states of the chain.

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(ii) Determine the period of each of the states of the chain.