



Subject: SRM 3

Chapter:

Category: Assignment 1 Solution

1.

i)	Start previous		Start morning		
	Day	1	2	3	4
	1	0.5	0	0	0.5
	2	0.25	0.5	0	0.25
	3	0.25	0.25	0.5	0
	4	0	0.25	0.25	0.5

[2]

- ii) If stationary distribution is $\pi = (\pi_1 \ \pi_2 \ \pi_3 \ \pi_4)$
Then $\pi A = \pi$ where A is the matrix in (i)

[0.5]

$$0.5 \pi_1 + 0.25 \pi_2 + 0.25 \pi_3 = \pi_1 \quad (I)$$

[0.5]

$$0.5 \pi_2 + 0.25 \pi_3 + 0.25 \pi_4 = \pi_2 \quad (II)$$

[0.5]

$$0.5 \pi_3 + 0.25 \pi_4 = \pi_3 \quad (III)$$

[0.5]

$$0.5 \pi_1 + 0.25 \pi_2 + 0.5 \pi_4 = \pi_4 \quad (IV)$$

[0.5]

$$\text{From (III), } \pi_3 = 0.5 \pi_4$$

[0.5]

$$\text{From (II), } \pi_2 = 0.75 \pi_4$$

[0.5]

$$\text{From (I), } \pi_1 = 0.625 \pi_4$$

[0.5]

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 = (0.625 + 0.75 + 0.5 + 1) \pi_4$$

[0.5]

Solving the above equation,

$$\pi_1 = 0.21739, \pi_2 = 0.26087, \pi_3 = 0.17391, \pi_4 = 0.34783$$

[0.5]

[Max 5]

- iii) Probability of restocking is 0.5 if in π_1 and 0.25 if in π_2
So long term rate = $0.5 * 0.21739 + 0.25 * 0.26087$

[1]

$$= 0.108695 + 0.06522 = 0.17391 \text{ per trading day}$$

[1]

[2]

iv) Probability of losing a sale is 0.25 if in π_1 [1]

So expected long term lost sales per day = $0.25 * 0.21739 = 0.05435$ [1]

[2]

v)	Start previous	Start morning		
	Day	2	3	4
	2	0.5	0	0.5
	3	0.25	0.5	0.25
	4	0.25	0.25	0.5

Let the stationary distribution be expressed as λ

Then $\lambda M = \lambda$ where M is the matrix above

$$\lambda_2 = 0.5 \lambda_2 + 0.25 \lambda_3 + 0.25 \lambda_4 \quad (A) \quad [0.5]$$

$$\lambda_3 = 0.5 \lambda_3 + 0.25 \lambda_4 \quad (B) \quad [0.5]$$

$$\lambda_4 = 0.5 \lambda_2 + 0.25 \lambda_3 + 0.5 \lambda_4 \quad (C) \quad [0.5]$$

From (B), $\lambda_3 = 0.5 \lambda_4$

From (A), $\lambda_2 = 0.75 \lambda_4$

Solving the equation $\lambda_2 + \lambda_3 + \lambda_4 = 1$, we get

$$\lambda_2 = 0.33333 \text{ or } 1/3 \quad [0.5]$$

$$\lambda_3 = 0.22222 \text{ or } 2/9 \quad [0.5]$$

$$\lambda_4 = 0.22222 / 0.5 = 0.44444 \text{ or } 4/9 \quad [0.5]$$

As no more than two vaccines sell per day, there are no lost sales.

Probability of restocking 0.5 if in λ_2 and 0.25 in $\lambda_3 = 0.5 * 0.33333 + 0.25 * 0.22222 = 0.22222$ [1]

[5]

vi) Restocking at two or more vaccines would not result in fewer lost sales than restocking at 1, because the probability of selling more than 2 vaccines is zero.

[1]

It would, however, result in more restocking charges than restocking at 1.

Therefore, it must result in lower profits than restocking at 1 so is not optimal.

[1]

[2]

vii) Costs if restock at zero vaccines: $0.17391C + 0.05435P$ [0.5]

Costs if restock at one vaccine: $0.22222C$

[0.5]

So should change restocking approach if $0.22222C < 0.17391C + 0.05435P$
i.e. $C < 1.1255P$

[1]

[2]

[20 Marks]

2.

Let S_n be the state of subscription started at time $t=0$.

$T(X_1=1 \text{ or } X_2=1 \mid X_0=0)$

$= T(X_1=1 \mid X_0=0) + T(X_1=0, X_2=1 \mid X_0=0)$

$= T(X_1=1 \mid X_0=0) + T(X_1=0 \mid X_0=0) \times T(X_1=2 \mid X_0=0)$

$= P_{0,0} + P_{0,00}XP_{1,01}$

$= 0.1 + 0.86 \times 0.11$

Expected number of cancellations by customers in next 2 years $= 500 \times (0.1 + 0.86 \times 0.11)$
 ~ 97

3.

- i) The score currently stands at 'Tie'. Whichever team wins the next point will move into a 'Lead'. If the team in 'Lead' wins the subsequent point as well, they would win the tournament. However, if the team in 'Lead' loses the next point, the score would be back at 'Tie'.

Since the probability of moving to the next state does not depend on the history prior to entering the state, Markov property holds.

The state space is defined as follows:

State	Description
T	Tie
L_{LSG}	LSG Leads
L_{GT}	GT Leads
G_{LSG}	LSG Wins
G_{GT}	GT Wins

[2.5]

ii)

$$a) \begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0 & 0.6 & 0 \\ 0.6 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[2.5]

iii)

After two points from the tie, the match would either be completed or be back to tie again.

The probability of returning to tie after two points is given by:

Probability of LSG winning the first point x Probability of GT winning the second point

+

Probability of GT winning the first point x Probability of LSG winning the second point

$$= 0.6 \times 0.4 + 0.4 \times 0.6 = 0.48$$

The number of such cycles (N) of returning to tie can be found by:

$$0.48^N = 1 - 0.9$$

Solving the above equation:

$$N = \ln(1-0.9)/\ln 0.48$$

$$= 3.14$$

Since the match can finish in cycles of two points, the required number of cycles is 4 i.e. 8 points.

[4]

iv)

After two points:

a. GT may have won the match with a probability of 0.16 ($= 0.4^2$); or

b. LSG may have won the match with a probability of 0.36 ($= 0.6^2$); or

c. It may have come back to tie with a probability of 0.48 (as calculated above).

Let LSG_T be the probability that LSG wins the match that is presently tied.

Let GT_T be the probability that GT wins the match that is presently tied.

We have:

$$GT_T = 0.16 + 0.48 \times GT_T$$

$$\text{Solving, } GT_T = 0.308$$

Probability that LSG eventually wins the match is 0.692 ($1 - GT_T$). This can be verified by:

$$LSG_T = 0.36 + 0.48 \times LSG_T$$

$$\text{Solving, } LSG_T = 0.692$$

[4]

- v) Probability of GT winning a point is 0.4. However, in order to win the game, GT would need to win at least two consecutive points. The probability of GT winning two consecutive points is lower than the probability of winning a point. At the same time, the probability of LSG winning the tournament would be more than the probability of GT winning it as the probability of LSG winning one point is more than that of GT winning a point.

[2]

[15 Marks]

4.

- i) Row vector π is called as stationary probability distribution for a Markov chain with transition matrix P if the following conditions hold for all j in S :

- $\pi = \pi P$ where π is row vector i.e. $\sum_{i=1}^n \pi_i = 1$.
- $\pi_i \geq 0$

[1.5]

In general a Markov chain need not have a stationary probability distribution, and if it exists it need not be unique.

[0.5]

(2)

- ii) Solution : (D).

(2)

iii)

$$\begin{aligned}
 P^2 &= \begin{pmatrix} 0.960 & 0.040 & 0.000 & 0.000 \\ 0.06 & 0.910 & 0.030 & 0.000 \\ 0.010 & 0.000 & 0.985 & 0.005 \\ 0.020 & 0.000 & 0.000 & 0.980 \end{pmatrix} \\
 & * \begin{pmatrix} 0.960 & 0.040 & 0.000 & 0.000 \\ 0.06 & 0.910 & 0.030 & 0.000 \\ 0.010 & 0.000 & 0.985 & 0.005 \\ 0.020 & 0.000 & 0.000 & 0.980 \end{pmatrix} \\
 &= \begin{pmatrix} 0.924 & 0.075 & 0.001 & 0.000 \\ 0.113 & 0.831 & 0.057 & 0.000 \\ 0.020 & 0.000 & 0.970 & 0.010 \\ 0.039 & 0.001 & 0.000 & 0.960 \end{pmatrix}
 \end{aligned}$$

$$q_k = P[X_0 = k], k=1,2,3,4$$

$$q'' = [0.5, 0.25, 0.15, 0.10]$$

$$q'P^2 = [0.497, 0.245, 0.160, 0.098]$$

Expected number of agents in each grade at the beginning of week 3 = [149, 74, 48, 29] [2]
[1]

iv)

$$(\pi_1 \ \pi_2, \ \pi_3, \ \pi_4) = (\pi_1 \ \pi_2, \ \pi_3, \ \pi_4) * \begin{pmatrix} 0.960 & 0.040 & 0.000 & 0.000 \\ 0.060 & 0.910 & 0.030 & 0.000 \\ 0.010 & 0.000 & 0.985 & 0.005 \\ 0.020 & 0.000 & 0.000 & 0.980 \end{pmatrix}$$

$$\pi_1 = 0.96 \pi_1 + 0.06\pi_2 + 0.01 \pi_3 + 0.02 \pi_4$$

$$\pi_2 = 0.04 \pi_1 + 0.91\pi_2$$

$$\pi_3 = 0.03\pi_2 + 0.985 \pi_3$$

$$\pi_4 = 0.005 \pi_3 + 0.98 \pi_4$$

$$0.04 \pi_1 - 0.09 \pi_2 = 0$$

$$0.03 \pi_2 - 0.005 \pi_3 = 0$$

$$0.02 \pi_4 - 0.005 \pi_3 = 0$$

$$\pi_3 = 4\pi_4$$

$$\pi_2 = 2\pi_4$$

$$\pi_1 = 4.5 \pi_4$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$$\pi_4 = 1/11.5 = 0.0870$$

$$\pi_1 = 4.5 \pi_4 = 0.3913$$

$$\pi_2 = 2 \pi_4 = 0.1739$$

$$\pi_3 = 4 \pi_4 = 0.3478$$

Thus, no of employees in steady states are $[\pi_1 \ \pi_2, \ \pi_3, \ \pi_4] * 300 = [117, 52, 104, 26]$ [2]

(5)

[14 Marks]

5. i) If each room is represented by the state, then the transition matrix P for this Markov chain is as follows:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} \quad [2]$$

ii) The chain is irreducible, because it is possible to go from any state to any other state. However, it is not aperiodic, because for any even n , $P_{6,1}^n$ will be zero and for any odd n , $P_{6,5}^n$ will also be zero. This means that there is no power of P that would have all its entries strictly positive.

[2]

iii) For P to be stationary,

$$\pi P = P$$

Perform matrix multiplication and show that πP is equal to P

[3]

iv) We find from π that the mean recurrence time (i.e. the expected time to return) for the room 1 is $1/\pi(1)=12$

[1]

v) Let, $\psi(i) = E(\text{number of steps to reach state 5} \mid X_0 = i).$

We have

$$\psi(5) = 0$$

$$\psi(6) = 1 + (1/2)\psi(5) + (1/2)\psi(4)$$

$$\psi(4) = 1 + (1/2)\psi(6) + (1/2)\psi(3)$$

$$\psi(3) = 1 + (1/4)\psi(1) + (1/4)\psi(2) + (1/4)\psi(4) + (1/4)\psi(5)$$

$$\psi(1) = 1 + \psi(3)$$

$$\psi(2) = 1 + \psi(3).$$

We solve and find $\psi(1) = 7$.

[1.5]

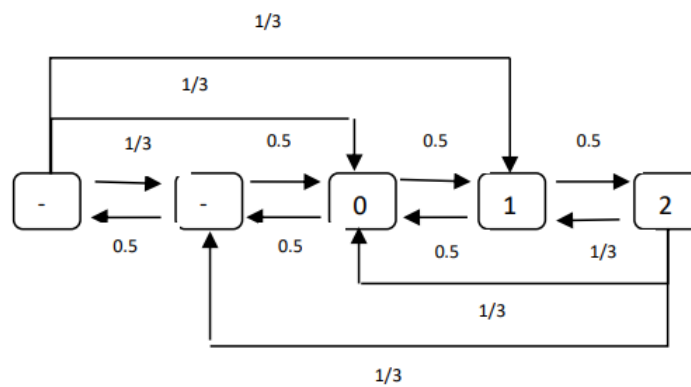
[1.5]

[3]

[11 Marks]

6.

i) Transition Diagram



[1.5]

	-2	-1	0	1	2
-2	0	1/3	1/3	1/3	0
-1	1/2	0	1/2	0	0
0	0	1/2	0	1/2	0
1	0	0	1/2	0	1/2
2	0	1/3	1/3	1/3	0

[1.5]

[3]

ii) $[\pi_{-2}, \pi_{-1}, \pi, \pi_1, \pi_2] = [\pi_{-2}, \pi_{-1}, \pi, \pi_1, \pi_2] P$

From the symmetry $\pi_{-2} = \pi_2, \pi_{-1} = \pi_1$, we have

[0.5]

$$\pi_0 = 1/3\pi_2 + 1/2\pi_1 + 1/2\pi_1 + 1/3\pi_2$$

$$\pi_1 = 1/3\pi_2 + 1/2\pi_0 + 1/3\pi_2$$

$$\pi_2 = 1/2\pi_1$$

$$\pi_0 + 2\pi_1 + 2\pi_2 = 1$$

[1]

From there we can get $\pi_0 = 4/3\pi_1$, and $\pi_1 = 3/13$ so

$$\pi = [3/26, 3/13, 4/13, 3/13, 3/26]$$

[1.5]

[3]

iii) This chain is irreducible since it is possible to move from each state to any other.

A periodic chain is one in which a state can only be revisited at multiples of some fixed number $d > 1$. State “-2” is aperiodic as it can be revisited after any number of steps. Also, since this chain is irreducible, all the states have the same periodicity. So the chain is aperiodic.

[2]
[8 Marks]

7.

i)

a) Data : RSRSSSSRRSRSSRSRRSSSSRRSSRRR

$$p_{rr} = 6/14 = 3/7$$

$$p_{rs} = 8/14 = 4/7$$

$$p_{sr} = 8/15$$

$$p_{ss} = 7/15$$

b)

30 th June	1 st Jul	2 nd Jul	3 rd Jul	
R	S	S	R	
	$p_{rs} = 8/14$	$p_{ss} = 7/15$	$p_{sr} = 8/15$	0.142222
R	S	R	R	
	$p_{rs} = 8/14$	$p_{sr} = 8/15$	$p_{rr} = 6/14$	0.130612
R	R	S	R	
	$p_{rr} = 6/14$	$p_{rs} = 8/14$	$p_{sr} = 8/15$	0.130612
R	R	R	R	
	$p_{rr} = 6/14$	$p_{rr} = 6/14$	$p_{rr} = 6/14$	0.078717

Total = 0.482164

ii)

a)

Here, $\mu = 3$

“Some policies “ means “1 or more policies” i.e 1 minus the “zero policies” probability:

$$P(X > 0) = 1 - P(x_0)$$

$$\text{Now, } P(X) = \frac{e^{-\mu} \mu^x}{x!}$$

$$\text{So, } P(x_0) = \frac{e^{-3} 3^0}{0!} = 4.9787 \times 10^{-2}$$

Therefore the probability of 1 or more policies is given by:

$$\begin{aligned} \text{Probability} &= P(X \geq 0) \\ &= 1 - P(x_0) \\ &= 1 - 4.9787 \times 10^{-2} \\ &= 0.95021 \end{aligned}$$

b)

The probability of selling 2 or more, but less than 5 policies is:

$$P(2 \leq X < 5) = P(x_2) + P(x_3) + P(x_4) \quad [1]$$

$$\begin{aligned} &= \frac{e^{-3} 3^2}{2!} + \frac{e^{-3} 3^3}{3!} + \frac{e^{-3} 3^4}{4!} \\ &= 0.61611 \end{aligned} \quad \begin{array}{l} [1.5] \\ [2.5] \end{array}$$

c)

$$\text{Average number of policies sold per day: } \frac{3}{5} = 0.6 \quad [1]$$

$$\text{So on a given day, } P(X) = \frac{e^{-0.6} 0.6^1}{1!} = 0.32929 \quad [1.5]$$

[2.5]

[12 Marks]