INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical & Risk Modelling - 3

Chapter:

Category: Assignment 1 Solution

1.

i)

Start previous Day	*Start morning* 1	2	3	4
1	0.5	0	0	0.5
2	0.25	0.5	0	0.25
3	0.25	0.25	0.5	0
4	0	0.25	0.25	0.5

[2]

ii) If stationary distribution is $\pi = (\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4)$

Then $\pi A = \pi$ where A is the matrix in (i)

[0.5]

$0.5\,\pi_1 + 0.25\,\pi_2 + 0.25\,\pi_3 = \pi_1$ (I) [0.5]

$0.5\,\pi_2 + 0.25\,\pi_3 + 0.25\,\pi_4 = \pi_2$ (II) [0.5]

$0.5\,\pi_3 + 0.25\,\pi_4 = \pi_3$ (III)

[0.5]

$0.5\,\pi_1 + 0.25\,\pi_2 + 0.5\,\pi_4 = \pi_4$ (IV)

[0.5]

From (III), $\pi_3 = 0.5\,\pi_4$

[0.5]

From (II), $\pi_2 = 0.75\,\pi_4$

[0.5]

From (I), $\pi_1 = 0.625\,\pi_4$

[0.5]

$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 = (0.625 + 0.75 + 0.5 + 1)\,\pi_4$

[0.5]

Solving the above equation,

$\pi_1 = 0.21739$, $\pi_2 = 0.26087$, $\pi_3 = 0.17391$, $\pi_4 = 0.34783$ [0.5]

[Max 5]

iii) Probability of restocking is 0.5 if in π_1 and 0.25 if in π_2

So long term rate = 0.5 * 0.21739 + 0.25 * 0.26087 [1]

= 0.108695 + 0.06522 = 0.17391 per trading day [1]

[2]

iv) Probability of losing a sale is 0.25 if in π_1 [1]

So expected long term lost sales per day = 0.25 * 0.21739 = 0.05435 [1]

[2]

v)

Start previous Day	*Start morning* 2	3	4
2	0.5	0	0.5
3	0.25	0.5	0.25
4	0.25	0.25	0.5

Let the stationary distribution be expressed as λ

Then $\lambda M = \lambda$ where M is the matrix above

$\lambda_2 = 0.5\ \lambda_2 + 0.25\ \lambda_3 + 0.25\ \lambda_4$ (A) [0.5]

$\lambda_3 = 0.5\ \lambda_3 + 0.25\ \lambda_4$ (B) [0.5]

$\lambda_4 = 0.5\ \lambda_2 + 0.25\ \lambda_3 + 0.5\ \lambda_4$ (C) [0.5]

From (B), $\lambda_3 = 0.5\ \lambda_4$

From (A), $\lambda_2 = 0.75\ \lambda_4$

Solving the equation $\lambda_2 + \lambda_3 + \lambda_4 = 1$, we get

$\lambda_2 = 0.33333$ or 1/3 [0.5]

$\lambda_3 = 0.22222$ or 2/9 [0.5]

$\lambda_4 = 0.22222 / 0.5 = 0.44444$ or 4/9 [0.5]

As no more than two vaccines sell per day, there are no lost sales.

Probability of restocking 0.5 if in λ_2 and 0.25 in λ_3 = 0.5*0.33333 + 0.25*0.22222 = 0.22222 [1]

[5]

vi) Restocking at two or more vaccines would not result in fewer lost sales than restocking at 1, because the probability of selling more than 2 vaccines is zero.

[1]

It would, however, result in more restocking charges than restocking at 1.
Therefore, it must result in lower profits than restocking at 1 so is not optimal. [1]

[2]

vii) Costs if restock at zero vaccines: $0.17391C + 0.05435P$ [0.5]

Costs if restock at one vaccine: $0.22222C$

[0.5]

So should change restocking approach if $0.22222C < 0.17391C + 0.05435P$
i.e. $C < 1.1255P$

[1]

[2]

[20 Marks]

2.

Let S_n be the state of subscription started at time t= 0.

T (X1=1 or X2 = 1 | X0 = 0)

= T (X1=1 | X0 = 0) + T (X1=0, X2 = 1 | X0 = 0)

= T (X1=1 | X0 = 0) + T (X1 = 0 | X0 = 0) X T (X1=2 | X0 = 0)

= $P_{0,0}$ + $P_{0,00}$XP1,01

= 0.1+0.86X0.11

Expected number of cancellations by customers in next 2 years = 500* (0.1+0.86X0.11) ~~ 97

3.

i) The score currently stands at 'Tie'. Whichever team wins the next point will move into a 'Lead'. If the team in 'Lead' wins the subsequent point as well, they would win the tournament. However, if the team in 'Lead' loses the next point, the score would be back at 'Tie'.
Since the probability of moving to the next state does not depend on the history prior to entering the state, Markov property holds.

The state space is defined as follows:

State	Description
T	Tie
L_{LSG}	LSG Leads
L_{GT}	GT Leads
G_{LSG}	LSG Wins
G_{GT}	GT Wins

[2.5]

ii)

a) $$\begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0 & 0.6 & 0 \\ 0.6 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[2.5]

iii)

After two points from the tie, the match would either be completed or be back to tie again.
The probability of returning to tie after two points is given by:
Probability of LSG winning the first point x Probability of GT winning the second point
+
Probability of GT winning the first point x Probability of LSG winning the second point
= 0.6 x 0.4 + 0.4 x 0.6 = 0.48
The number of such cycles (N) of returning to tie can be found by:
$0.48^N = 1 - 0.9$
Solving the above equation:
N= ln(1-0.9)/ln 0.48
= 3.14
Since the match can finish in cycles of two points, the required number of cycles is 4 i.e. 8 points. [4]

iv)

After two points:
a. GT may have won the match with a probability of 0.16 (= 0.4^2); or
b. LSG may have won the match with a probability of 0.36 (= 0.6^2); or
c. It may have come back to tie with a probability of 0.48 (as calculated above).

Let LSG_T be the probability that LSG wins the match that is presently tied.
Let GT_T be the probability that GT wins the match that is presently tied.

We have:
$GT_T = 0.16 + 0.48 \times GT_T$

Solving, $GT_T = 0.308$

Probability that LSG eventually wins the match is 0.692 ($1-GT_T$). This can be verified by:
$LSG_T = 0.36 + 0.48 \times LSG_T$

Solving, $LSG_T = 0.692$ [4]

v) Probability of GT winning a point is 0.4. However, in order to win the game, GT would need to win at least two consecutive points. The probability of GT winning two consecutive points is lower than the probability of winning a point. At the same time, the probability of LSG winning the tournament would be more than the probability of GT winning it as the probability of LSG winning one point is more than that of GT winning a point. [2]

[15 Marks]

4.

i) Row vector Pi is called as stationary probability distribution for a Markov chain with transition matrix P if the following conditions hold for all j in S:

- $\pi = \pi P$ where π is row vector i.e.-Σ-
- $\pi_i >= 0$
- Σ =1

[1.5]

In general a Markov chain need not have a stationary probability distribution, and if it exists it need not be unique.

[0.5]

(2)

ii) Solution : (D). (2)

iii)

$$P^2 = \begin{pmatrix} 0.960 & 0.040 & 0.000 & 0.000 \\ 0.06 & 0.910 & 0.030 & 0.000 \\ 0.010 & 0.000 & 0.985 & 0.005 \\ 0.020 & 0.000 & 0.000 & 0.980 \end{pmatrix}$$

$$* \begin{pmatrix} 0.960 & 0.040 & 0.000 & 0.000 \\ 0.06 & 0.910 & 0.030 & 0.000 \\ 0.010 & 0.000 & 0.985 & 0.005 \\ 0.020 & 0.000 & 0.000 & 0.980 \end{pmatrix}$$

$$= \begin{pmatrix} 0.924 & 0.075 & 0.001 & 0.000 \\ 0.113 & 0.831 & 0.057 & 0.000 \\ 0.020 & 0.000 & 0.970 & 0.010 \\ 0.039 & 0.001 & 0.000 & 0.960 \end{pmatrix}$$

[2]

$q_k = P[X_0 = k]$, k=1,2,3,4
q'' = [0.5,0.25,0.15,0.10]

q'P^2 = [0.497,0.245,0.160,0.098]

[2]

Expected number if agents in each grade at the beginning of week 3 = [149, 74, 48, 29] [1]

iv)

$$(\pi_1\ \pi_2, \pi_3, \pi_4) = \quad (\pi_1\ \pi_2, \pi_3, \pi_4) * \begin{pmatrix} 0.960 & 0.040 & 0.000 & 0.000 \\ 0.060 & 0.910 & 0.030 & 0.000 \\ 0.010 & 0.000 & 0.985 & 0.005 \\ 0.020 & 0.000 & 0.000 & 0.980 \end{pmatrix}$$

$\Pi_1 = 0.96\ \Pi_1 + 0.06\Pi_2 + 0.01\ \Pi_3 + 0.02\ \Pi_4$
$\Pi_2 = 0.04\ \Pi_1 + 0.91\Pi_2$
$\Pi_3 = 0.03\Pi_2 + 0.985\ \Pi_3$
$\Pi_4 = 0.005\ \Pi_3 + 0.98\ \Pi_4$

$0.04\ \Pi_1 - 0.09\ \Pi_2 = 0$

$0.03\ \Pi_2 - 0.005\ \Pi_3 = 0$

$0.02\ \Pi_4 - 0.005\ \Pi_3 = 0$

$\Pi_3 = 4\Pi_4$
$\Pi_2 = 2\Pi_4$

$\Pi_1 = 4.5\ \Pi_4$

$\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 = 1$

$\Pi_4 = 1/11.5 = 0.0870$
$\Pi_1 = 4.5\ \Pi_4 = 0.3913$
$\Pi_2 = 2\ \Pi_4 = 0.1739$
$\Pi_3 = 4\ \Pi_4 = 0.3478$

Thus, no of employees in steady states are $[\pi_1\ \pi_2, \pi_3, \pi_4] * 300$ = [117,52,104,26] [2]

(5)

[14 Marks]

5.

$P^2AA = 0.5625 \Rightarrow a^2 = 0.5625 \Rightarrow a = 0.75$

Rows of transition matrix must sum to 1.
So, $a + c = 1$
and $c = 0.25$

$P^2AB = 0.125 \Rightarrow ch = 0.125 \Rightarrow h = 0.5$

$h + i = 1$
so $i=0.5$

$P^2CC = 0.4 \Rightarrow f \times 0.5 + 0.5^2 = 0.4 \Rightarrow f = 0.3$

$P^2BA = 0.475 \Rightarrow d\,(0.75 + e) = 0.475$
Rows sum to 1 so, $d + e = 0.7$
Substitute for e:
$d(1.45 - d) = 0.475 \Rightarrow d^2 - 1.45d + 0.475 = 0$
Solving using standard quadratic formula:
$d = 0.95$ or 0.5
0.95 is not possible because e would need to be negative

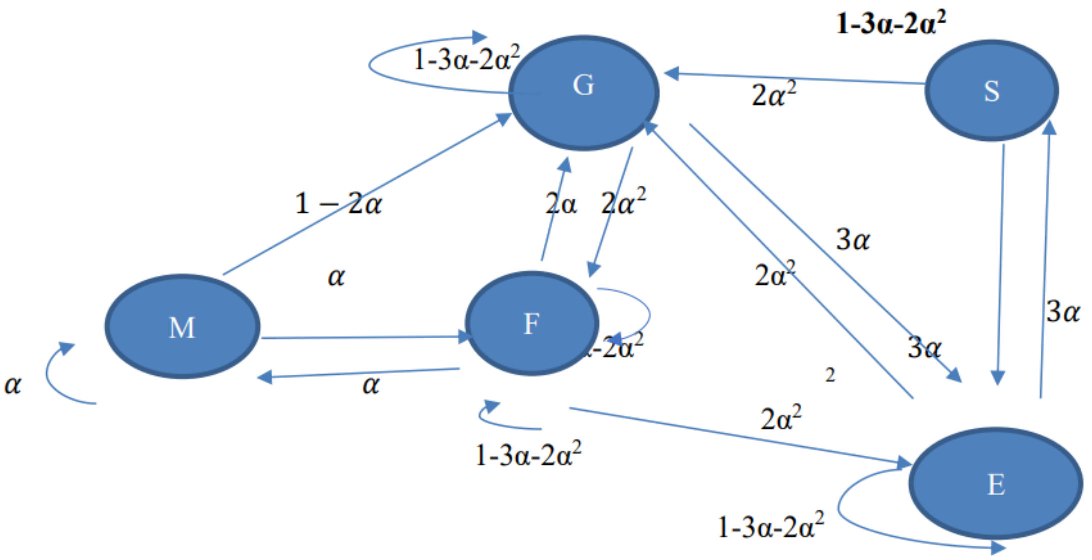
So $d = 0.5$ and $e = 0.2$

Transition matrix is:

0.75	0	0.25
0.5	0.2.	0.3
0	0.5	0.5

6.

i)



ii) The transition matrix will be valid if the entries in each row add up to 1 (which is true) and each row entry lies in the range [0,1]

So, $0 \leq \alpha \leq 1$;

$0 \leq 1 - 2\alpha \leq 1$ => $0 \leq 2\alpha \leq 1$ => $0 \leq \alpha \leq \frac{1}{2}$

$0 \leq 3\alpha \leq 1$ => $0 \leq \alpha \leq \frac{1}{3}$

$0 \leq$ 1-3α-2α^2 ≤ 1

Since $\alpha \geq 0$, it automatically implies that 1-3α-2 α^2 $\leq$1

We need to find the values of α for which $0 \leq 1-3\alpha-2\alpha^2$

The equation $0 = 1-3\alpha-2\alpha^2$ => $2\alpha^2+3\alpha-1 = 0$

- ⇨ $\alpha = \frac{(-3\pm\sqrt{17})}{4}$ = 0.280776 or -1.78078
- ⇨ $-1.78078 \leq \alpha \leq 0.280776$

Putting these together, we can see that all of the conditions

$0 \leq \alpha \leq 1$

$0 \leq \alpha \leq \frac{1}{2}$

$0 \leq \alpha \leq \frac{1}{3}$

$-1.78078 \leq \alpha \leq .280776$

So we must have

$0 \leq \alpha \leq 0.280776$

iii) The chain is irreducible since every state can be eventually reached from every other state.

If $\alpha < 0.280766$ every state has an arrow to itself, so every state is aperiodic.

When $\alpha = 0.280766$, none of the states has an arrow to itself. However, return to each of these states is possible in 2 or 3 or 4 steps. So, each state is aperiodic. So the chain is aperiodic.

iv) When $\alpha = 0.1$ the above transition probability matrix becomes

$$\begin{bmatrix} 0.1 & 0.1 & 0.8 & 0 & 0 \\ 0.1 & 0.68 & 0.2 & 0.02 & 0 \\ 0 & 0.02 & 0.68 & 0.3 & 0 \\ 0 & 0 & 0.02 & 0.68 & 0.3 \\ 0 & 0 & 0.02 & 0.3 & 0.68 \end{bmatrix}$$

The stationary distribution is the vector of probabilities **π**P = **π,** where P is the transition matrix above. Writing out the equations, we have

$0.1\pi_1+0.1\pi_2 = \pi_1$-------------------------(1)

$0.1\pi_1+0.68\pi_2+0.02\pi_3 = \pi_2$------------------(2)

$0.8\pi_1+0.2\pi_2+0.68\pi_3 + 0.02\pi_4 + 0.02\pi_5 = \pi_3$------------------(3)

$0.02\pi_2+0.3\pi_3 + 0.68\pi_4 + 0.3\pi_5 = \pi_4$------------------(4)

$0.3\pi_4 + 0.68\pi_5 = \pi_5$------------------(5)

We can discard equation (3) and replace with

$\pi_1+\pi_2+\pi_3 + \pi_4 + \pi_5 = 1$

From equation (1), $\pi_2 = 9\pi_1$ ----------------------(6)

Substituting in (2) ,

$0.1\pi_1 + 0.02\pi_3 = 0.32 * 9\pi_1$
$0.02\pi_3 = 2.78\pi_1$
$\pi_3 = 139\pi_1$ ------------------------------(7)

First we solve equations (4) and (5) by eliminating π_5.

From (5) we get, $0.3\pi_4 = 0.32\,\pi_5$

$\pi_5 = 0.3/0.32\pi_4$

Substituting in (4) we get,

$0.02\pi_2 + 0.3\pi_3 + 0.68\pi_4 + 0.3 * 0.3/0.32\pi_4 = \pi_4$

$0.02\pi_2 + 0.3\pi_3 = \pi_4 * (1 - 0.68 - 0.09/0.32) = 0.03875\,\pi_4$

Substituting for π_2 *and* π_3 from equations (6) and (7) in this equation we get

$.02*9\pi_1 + 0.3*139\,\pi_1 = 0.03875\,\pi_4$

$\pi_4 = 1080.774\pi_1$

From (5) $0.32\pi_5 = 0.3\pi_4$

$$\pi_5 = \frac{0.3}{0.32} * 1080.774\pi_1 = 1013.226\pi_1$$

Using the relationship $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$ we get

$(1+9+139+1080.774+1013.226)\,\pi_1 = 1$

$\pi_1 = 0.000446$

$\pi_5 = 1013.226 * 0.000446 = 0.451728$

7.

i)

a) Data : RSRRSSSSRRSRSSRSRSRRSSSRRSSRRR

$p_{rr} = 6/14 = 3/7$

$p_{rs} = 8/14 = 4/7$

$p_{sr} = 8/15$

$p_{ss} = 7/15$

b)

30th June	1st Jul	2nd Jul	3rd Jul	
R	S	S	R	
	p_{rs} = 8/14	p_{ss} = 7/15	p_{sr} = 8/15	0.142222
R	S	R	R	
	p_{rs} = 8/14	p_{sr} = 8/15	p_{rr} = 6/14	0.130612
R	R	S	R	
	p_{rr} = 6/14	p_{rs} = 8/14	p_{sr} = 8/15	0.130612
R	R	R	R	
	p_{rr} = 6/14	p_{rr} = 6/14	p_{rr} = 6/14	0.078717

Total = 0.482164

ii)

a)

Here, $\mu = 3$

"Some policies " means "1 or more policies" i.e 1 minus the "zero policies" probability:

$$P(X > 0) = 1 - P(x_0)$$

Now, $P(X) = \frac{e^{-\mu}\mu^x}{x!}$

So, $P(x_0) = \frac{e^{-3}3^0}{0!}$ = 4.9787 x 10^{-2}

Therefore the probability of 1 or more policies is given by:

Probability = $P(X \geq 0)$

= $1 - P(x_0)$

= 1- 4.9787 x 10^{-2}

= 0.95021

b)

The probability of selling 2 or more, but less than 5 policies is:

$P(2 \leq X < 5) = P(x_2) + P(x_3) + P(x_4)$ [1]

$$= \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!}$$

= 0.61611 [1.5]

[2.5]

c)

Average number of policies sold per day: $\frac{3}{5}$ = 0.6 [1]

So on a given day, $P(X) = \frac{e^{-0.6}0.6^1}{1!}$ = 0.32929 [1.5]

[2.5]

[12 Marks]

8.

i)

Chain is irreducible since every state can be reached from every other state. All states will have a period of 1

ii)

Claim is irreducible and aperiodic.

Long-term bonus =∑ (i = 1 to 4) $d_i * Л_i$

Where d_i = Yearly bonus in state 'i'.

$0.2 Л_1 + 0 + 0 + 0.4 Л_4 = Л_1$

$0.8 Л_1 + 0.7 Л_2 + 0 + 0 = Л_2$

$0 + 0.3 Л_2 + 0.4 Л_3 + 0 = Л_3$

$0 + 0 + 0.6 Л_3 + 0.6 Л_4 = Л_4$

$Л_1 + Л_2 + Л_3 + Л_4 = 1$

Solving: $Л_1 = 1/7$

$Л_2 = 8/21$

$Л_3 = 4/21$

$Л_4 = 2/7$

∴ Long-term yearly bonus = 9x1/7 + 5X8/21+0X4/21+3X2/7

= 4.0476