INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical & Risk Modelling - 3

Chapter:

Category: Assignment 2 Solutions

1.

i)

	O	I
O	-0.25	0.25
I	0.75	-0.75

ii)

Transition probabilities do not depend on history. Prior to coming in & Out form.

Therefore, it is Markov. Continuous time operation & state space is discrete.

iii)

$dP_{OO}(t)/dt = 0.75 \times P_{OI}(t) - 0.25 \times dP_{OO}(t)$

$P_{OI}(t) + P_{OO}(t) = 1$

Substituting:

$dP_{OO}(t)/dt + P_{OO}(t) = 0.75$

$d\, e^t \times P_{OO}(t)/dt = 0.75 \times e^t$

$\rightarrow e^t \times P_{OO}(t) = 0.75 \times e^t + k$

$P_{OO}(0) = 1 \rightarrow k = 0.25$

$P_{OO}(t) = 0.75 + 0.25 \times e^{-t}$

2.

i) The Chapman – Kolmogrov equations are

$$P_{ij}(s,t) = \sum_{k \in s} P_{ik}(s,u) P_{kj}(u,t)$$

To obtain the forward equations we differentiate with respect to t and evaluate at u=t;

$$\frac{\partial}{\partial t} P_{ij}(s,t) = \sum_{k \in s} \left[P_{ik}(s,u) \left(\frac{\partial}{\partial s} P_{kj}(u,t) \right) \right]_{u=t} = \sum P_{ik}(s,t) \mu_{kj}(t)$$

Similarly the backward equations are obtained by differentiating with respect to s and setting u=s ;

$$\frac{\partial}{\partial s} P_{ij}(s,t) = \sum_{k \in s} \left[\left(\frac{\partial}{\partial s} P_{ik}(s,u) \right) P_{kj}(u,t) \right]_{u=s} = -\sum_{k \in s} \mu_{ik}(s) P_{kj}(s,t)$$

We now need to explain where the minus sign in the RHS comes from.
The definition of the transition rates is such that;

$$P_{ik}(s,s+h) = \delta_{ik} + h\mu_{ik}(s) + o(h)$$

Or equivalently;

$$P_{ik}(s-h,s) = \delta_{ik} + h\mu_{ik}(s-h) + o(h)$$

Rearranging this gives:

$$\mu_{ik}(s-h) = \frac{P_{ik}(s-h,s) - \delta_{ik} - o(h)}{h}$$

Now taking the limit of both sides as h->0 and nothing that $P_{ik}(s,s)=\delta_{ik}$, we get

$$\mu_{ik}(s) = -\lim_{h\to 0}\frac{P_{ik}(s-h,s) - P_{ik}(s,s) - o(h)}{-h} = -\left[\frac{\partial}{\partial s}P_{ik}(s,t)\right]_{t=s}$$

ii) Kolmogorov's forward differential equation

The matrix form of the forward differential equation is:

$\partial/\partial t\, P(s,t) = -\, P(s,t)\, A(t)$

Since this model is time inhomogeneous and we are asked for the forward differential equation, we are differentiating with respect to t.

For this model:

$\partial/\partial t P_{PS}(s,t) = -[P_{PP}(s,t)\ 0.15t + P_{PA}(s,t)\ 0.1t - P_{PS}(s,t) 0.01t]$ [2]
and:
$\partial/\partial t\, P_{SS}(s,t) = -[P_{SS}(s,t)\ (-0.01t)] = 0.01t\ P_{SS}(s,t)$ ....(1)

[2]

iii) From equation 1 above,
Separating the variables gives:

$\partial/\partial t\, P_{SS}(s,t) / P_{SS}(s,t) = 0.01t$

and changing the variables from t to u:

$\partial/\partial u \ln P_{SS}(s,u) = 0.01t$

Integrating both sides with respect to u between the limits u = s and u = t, we get:

$[\ln P_{SS}(s,u)]^t_s$ = integrate (t,s) 0.01u du = $[0.005u^2]^t_s$
i.e. $\ln P_{SS}(s,s) - \ln P_{SS}(s,t) = 0.005\,(t^2 - s^2)$

However, since $P_{SS}(s,s) = 1$ and ln 1 = 0, we have,
$-\ln P_{SS}(s,t) = 0.005\,(t^2 - s^2)$

The expression above can be rearranged to give:
$P_{SS}(s,t) = e^{-0.005\,(t2 - s2)}$

Now, substituting the value of t = 5 and s = 3, we get:
$P_{SS}(s,t) = e^{-0.08} = 92.31\%$

3. i) Model fitting: this occurs after the family of model has been decided and concerns the estimation of the values of parameters. The set of parameters to be estimated is determined by the choice of model family. Model verification: once the model has been fitted we need to check that the fitted process resembles what has been observed. Generally we produce simulations of the process, using the estimated parameter values, and compare them with the observations.

ii) The parameters required to be estimated to model this as Markov processes are:

- Rate of leaving state *i*, Λ_i for each *i*,
- the jump chain transition probabilities, r_{ij} for $j \neq i$, where r_{ij} is the conditional probability that the next transition takes the chain to state *j* given that it is now in state *i*.

Assumptions of the Markov model for determination of these parameters:

- Duration of holding time in state *i* has exponential distribution with parameter determined only by *i* and is independent of anything that happened before the current arrival in state *i*,
- Destination of the next jump after leaving state *i* is independent of the holding time in state *i* and of anything that happened before the chain arrived in state *i*.

iii) The average duration of stay in any particular state i is given by $\frac{1}{\lambda_i}$

State	$1/\lambda_i$
1	6
2	40
3	30

The transition probabilities are as follows:

$$r_{12} = \frac{3}{8}, r_{13} = \frac{5}{8}, r_{21} = \frac{1}{4}, r_{23} = \frac{3}{4}, r_{31} = \frac{7}{8} \text{ and } r_{32} = \frac{1}{8}$$

Thus the generator matrix is

$$\begin{pmatrix} -1/6 & 3/48 & 5/48 \\ 1/160 & -1/40 & 3/160 \\ 7/240 & 1/240 & -1/30 \end{pmatrix}$$

4. i) A Markov jump process is a continuous-time Markov process with a discrete state space. For a process to be Markov, the future development of the process must depend only on its current state. This is the case here, as the future of the process depends only on the number of consensus for the current transaction. The number of consensus for the current transaction also has a discrete state space {0, 1, 2, 3}. (Note: that immediately after the 4th consensus, the next transaction from the queue is shared with the blockchain network nodes, so fourth consensus is not required for modelling purpose and all nodes must now work on new transaction to solve its cryptographic problem.)

ii) The generator matrix *A* is

$$\begin{vmatrix} -\beta & \beta & 0 & 0 \\ 0 & -\beta & \beta & 0 \\ 0 & 0 & -\beta & \beta \\ \beta & 0 & 0 & -\beta \end{vmatrix}$$

iii) Kolmogorov's forward equation can be written in compact form as

$$\frac{d}{dt}P(t) = P(t)A$$

Which are , for j =0

$$\frac{d}{dt}p_{i0}(t) = \beta p_{i3}(t) - \beta p_{i0}(t)$$

And for j = 1,2,3

$$\frac{d}{dt}p_{ij}(t) = \beta p_{i,j-1}(t) - \beta p_{ij}(t)$$

iv) Since the waiting times under a Poisson process are exponential, the expected waiting time between the arrival of consensus for the current transaction is 1/β minutes. Successive waiting times are independent, therefore the expected waiting time for a node for next transaction is

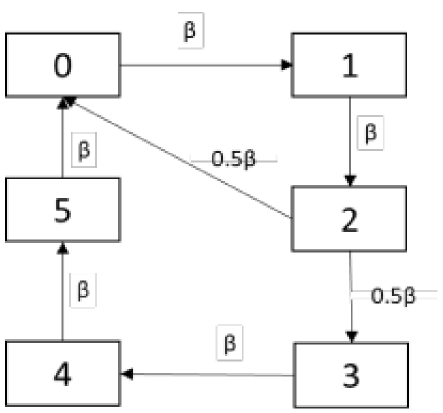
$$E(t) = \sum_{i=0}^{3} p_i \frac{3-i}{\beta},$$

where $p_i$is the probability that the transaction has already received "i" consensus when the

new consensus arrives.
Since the $p_i s$ are all equal for *i* = 0, 1, 2, 3

$$E(t) = 0.25\left(\frac{3}{\beta} + \frac{2}{\beta} + \frac{1}{\beta} + \frac{0}{\beta}\right) = \frac{3}{2\beta} \text{minutes}$$

v) The diagram and the transition matrix, *P*, is



$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

vi) The expected waiting time if the current transaction is less than 100k is:

$$E(t|3 - less\ than\ 100k\ Transaction) = \sum_{i=0}^{2} p_i \frac{2-i}{\beta}$$

$$= \frac{1}{3}\left(\frac{2}{\beta}+\frac{1}{\beta}+\frac{0}{\beta}\right) = \frac{1}{\beta}$$

The expected waiting time if the current transaction is more than 100k is:

$$E(t|6 - more\ than\ 100k\ Transaction) = \sum_{i=0}^{5} p_i \frac{5-i}{\beta}$$

$$= \frac{1}{6}\left(\frac{5}{\beta}+\frac{4}{\beta}+\frac{3}{\beta}+\frac{2}{\beta}+\frac{1}{\beta}+\frac{0}{\beta}\right) = \frac{5}{2\beta}$$

But 6-consensus transaction must expect to wait 2 times as long for addition to blockchain than 3-consensus transaction takes.

So when a consensus arrives for the current transaction, 2/3 (6/9) of the time the transaction at the front of the queue will be a transaction with value greater than 100K and only of the 1/3(3/9) time will is be a transaction with value less than 100k.

So the overall expected waiting time in minutes is

1/3*(E(t| 3 consensus transaction)) + 2/3*(E(t| 6 consensus transaction))

$$= \frac{1}{3} * \frac{1}{\beta} + \frac{2}{3} * \frac{5}{2\beta} = \frac{2}{\beta}$$

As this is longer than 3/2β, the time to add transaction to the blockchain has increased.

5.

i) $P_{10}(t+h) = (1- \rho h).P_{10}(t) + \sigma h P_{11}(t)$

$= d/dtP10\ (t) = - \rho\ P_{10}(t) + \sigma h P_{11}(t)$

And

$P_{11}\ (t+h) = \rho h\ P_{10}(t) + (\ 1- \sigma h\ )P_{11}(t)$

$= d/dtP_{11}(t) = \rho P_{10}(t) - \sigma P_{11}(t)$ ..... eq (2)

[(2) also follows from the fact that $P_{10}\ (t)+ P_{11}(t) =1$]

ii) $P_{10}(t)+ P_{11}(t) =1$

so from (1) $d/dt\ P_{10}(t) + (\sigma + \rho)\ .\ P_{10}(t) = \sigma$

$d/dtExp\ ((\sigma + \rho)t)\ P_{10}(t) = \sigma * Exp\ ((\sigma + \rho)t) +C$

$P_{10}(t) =)= \sigma /(\sigma + \rho)*\ \exp\ (\sigma + \rho)\ *t) +C* \exp - (\sigma + \rho)\ *t$

$P_{10}(t) =)= \sigma /(\sigma + \rho)*\ (\ 1- \exp - (\sigma + \rho)\ *t)$ since P10(0) = 0.

Therefore $P11(t) = 1 - \sigma /(\sigma + \rho)\ )*\ (\ 1- \exp - (\sigma + \rho)\ *t) = \frac{(\rho + \sigma \exp\ (-(\sigma + \rho)*t)}{\sigma + \rho}$

iii)

a) The generator matrix is now

$$\begin{matrix} -\rho & \rho & 0 \\ \sigma & -(\sigma+\rho) & \rho \\ 0 & 2\sigma & -2\sigma \end{matrix}$$

b) Hence the Forward Equations are

$d/dt(\ P_0(t) = - \rho\ *p_0(t) + \sigma\ p_1(t)$

$d/dt(P_1(t)\ ) = \rho\ p_0 - (\sigma + \rho\ )\ p_{1(t)} + 2\ \sigma p_2(t)$

$d/dt(P_2(t)\ ) = \rho\ p_1(t) - 2\ \sigma p_2(t)$

c) Simply substituting in the suggested values gives the required result.

d) The implication is that the given distribution is stationary. By the standard properties of Markov processes, it follows that it is the equilibrium distribution, so that the long-term probabilities of being in each of the three states are known.

6.

i) Let Z_n= time between arrival of the n^{th} and $(n – 1)^{th}$ clients.
Then Z_n's are i.i.d. exponential random variables with mean 1 / λ i.e. $E[Z_n]=\frac{1}{\lambda}$
Let T_n = arrival time of the n^{th} passenger = $\sum_{i=1}^{n} z_n$

$$\boldsymbol{E[T_n] = E\left[\sum_{i=1}^{n} z_n\right] = \sum_{i=1}^{n} E[z_n] = \frac{n}{\lambda}}$$

The expected waiting time until the first chopper takes off is

$\boldsymbol{E[T_3]} = \frac{\mathbf{3}}{\mathbf{[1/15]}}$ = 45 minutes

ii) Let X(t) be the Poisson process with mean λ*t. Note that P(X(t)) = k =**$(e^{-\lambda t}*(\lambda t)^k)/k!$,**
For k= 1, 2, 3….
We have
P = P[No helicopter takes off in the first 2 hours]
=P[At most 2 passengers in first 120 mins}]
=P[{X(t) <= over (0,120)}]
=P[{X(120) <= 2]
=P[X(120) = 0] + P[X(120) = 1] + P[X(120) = 2]
$= e^{-\frac{120}{15}} + \left(\frac{120}{15}\right) e^{-\frac{120}{15}} + \frac{1}{2}\left(\frac{120}{15}\right)^2 e^{-\left(\frac{120}{15}\right)}$
= 0.01375
= 1.375%

iii) In order to ensure that the operator flies at least 3 trips in next 3 hours before the weather conditions worsen, there should be at least 9 passengers.

(1)

Hence, the probability that at least 9 clients would arrive is

P = P[{At least 9 passengers arrive in 180 mins}]

= 1 – P[{At most 8 clients arrive in 180 mins }]

= 1 – P[{X(180) <= 8]

= 1 - $\sum_{k=0}^{k=8} P\,[X(180) = k]$

= 1 – P[X(180) = 0] – P[X(180) = 1] – P[X(180) = 2] – P[X(180) = 3] – P[X(180) = 4] – P[X(180) = 5] – P[X(180) = 6] – P[X(180) = 7] – P[X(180) = 8]

$$= 1 - e^{-\frac{180}{15}}\left[\sum_{0}^{8}\left(\frac{180}{15}\right)^{k} * \frac{1}{k!}\right]$$

(1)

= 1 – 0.15503

= 84.5% (1)

[3]

[7 Marks]

7.

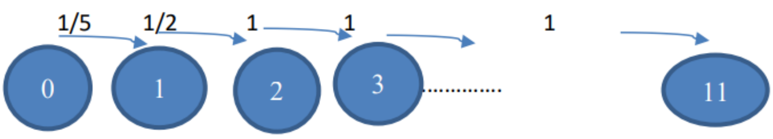
i) This is a 12-state Markov jump process. The states are (1) 0 point, (2) 1 points, (3) 2 points,........(12) 11 points.

The Markov assumption is that the probability of jumping to any particular state depends only on knowing the current state that is occupied and the transition rates between states are constant over time.

ii) Let N(t) be the number of points scored up to time t. Then the state space N(t) is the set {0,1,2,3......................11}

Since the game starts at point 0 we have N(0)=0.

The transition diagram is



The required probability is

P(N(6)> 2) = 1- P(N(6) =0)- P(N(6) =1)- P(N(6) =2)

First, we have:

P(N(6) =0) = $e^{-6*1/5}$ = 0.301194,

<u>Calculating P(N(6) =1)</u>

Forward equation for P(N(t) =1) = $P_{01}(t)$ is

$\frac{d}{dt}P_{01}(t) = \frac{1}{5}P_{00}(t) - \frac{1}{2}P_{01}(t)$

Solving by integrating factor method, by first writing in the form:

$$\frac{d}{dt}P_{01}(t) + \frac{1}{2}P_{01}(t) = \frac{1}{5}P_{00}(t)$$

We have $P_{00}(t) =$ P(N(t) =0) $e^{-t/5}$

Substituting for $P_{00}(t)$, the differential equation becomes

$\frac{d}{dt}P_{01}(t) + \frac{1}{2}P_{01}(t) = \frac{1}{5} e^{-t/5}$

Multiplying by the integrating factor $e^{t/2}$ gives:

$e^{t/2} * \frac{d}{dt}P_{01}(t) + e^{t/2} * \frac{1}{2}P_{01}(t) = \frac{1}{5} e^{3t/10}$

LHS can be written as:

$\frac{d}{dt}(e^{t/2} * P_{01}(t))$

Integrating, we get,

$e^{t/2} * P_{01}(t) = 1/5 * 10/3 * e^{3t/10} + C$

$e^{t/2} * P_{01}(t) = 2/3 * e^{3t/10} + C$
When t= 0: $P_{01}(0) = 0$
Substituting t=0,

0 = 2/3 + C
C= -2/3

$e^{t/2} * P_{01}(t) = 2/3 * e^{3t/10} - 2/3$

$P_{01}(t) = 2/3 * e^{-t/5} - 2/3\ e^{-t/2}$

P(N(6) =1) = $P_{01}(6) = 2/3 * (e^{-6/5} - e^{-6/2}) = 0.167605$

Calculating P(N(6) =2)

Forward equation for P(N(t) =2) = $P_{02}(t)$ is

$$\frac{d}{dt}P_{02}(t) = P_{00}(t)\mu_{02} + P_{01}(t)\mu_{12} + P_{02}(t)\mu_{22}$$

Using the transition rates $\mu_{02} = 0$, $\mu_{12} = \frac{1}{2}$, $\mu_{22} = -1$

$$\frac{d}{dt}P_{02}(t) = 1/2 * P_{01}(t) - P_{02}(t)$$

$$\frac{d}{dt}P_{02}(t) + P_{02}(t) = \frac{1}{2}P_{01}(t)$$

Substituting for $P_{01}(t)$,

$$\frac{d}{dt}P_{02}(t) + P_{02}(t) = \frac{1}{2} * \{2/3 * e^{-t/5} - 2/3\, e^{-t/2}\}$$

$$= 1/3 * \{e^{-t/5} - e^{-t/2}\}$$

Multiplying by the integrating factor e^t, we get

$$\frac{d}{dt}\{P_{02}(t)e^t\} = 1/3 * \{e^{4t/5} - e^{t/2}\}$$

Integrating,

$$P_{02}(t)e^t = 1/3 * \{5/4 * e^{4t/5} - 2 * e^{t/2}\} + C$$

Since , $P_{02}(0) = 0$

$$0 = \frac{1}{3} * \left\{\frac{5}{4} - 2\right\} + C$$

0= -1/4 + C

C= ¼

$$P_{02}(t)e^t = 1/3 * \{5/4 * e^{4t/5} - 2 * e^{t/2}\} + ¼$$

$$P_{02}(t) = 1/3 * \{5/4 * e^{-t/5} - 2 * e^{-t/2}\} + ¼ * e^{-t}$$

$$P(N(6) = 2) = P_{02}(6) = 1/3 * \{5/4 * e^{-6/5} - 2 * e^{-3}\} + ¼ * e^{-6}$$

$$= 0.092926$$

Probability of scoring more than two points in next 6 minutes

= 1- P(N(6) =0)- P(N(6) =1)- P(N(6) =2)

= 1-0.301194 -0.167605-0.092926

=0.438275

iii) For i= 0,1,2,……10 the expected holding time in state i is $1/\lambda_i$ where λ_i is the total force out of State i.

State i must be followed by state i+1 . So the expected time to complete the game is

$$\sum_{i=0}^{10} 1/\lambda i = 5+2+(1*9) = 16 \text{ minutes}$$

8.

i) Claims of type Silver occur according to a Poisson process with a mean of 50%* 4 = 2 per day.

So the waiting time until the first claim of type Silver has an exp(2) distribution and the expected waiting time is 1/2 days.

ii) Let N(t) denote the number of claims during the interval [0,t]. Then:

P[N(2)>=9| N(1)=7]
= P[N(2)- N(1)>=2| N(1)- N(0)=7]
= P[N(2)- N(1)>=2]

Since N(0)=0 and assuming number of claims in non-overlapping time intervals are independent.

N(2)-N(1) ˜ *Poisson*(4), so:

P(N(2)>=9| N(1)=7) =P(*Poisson*(4)>=2)
=1- P(*Poisson*(4)<=1)
= 1- exp(-4) (1+4/1!)
=1- exp(-4)*5
= 1- 0.018316*5
= 0.908422

iii) Let $N_G(t)$ denote the number of claims of Gold Type in the interval [0,t].

$P(N_G(1) >=5, N_G(2))>=7)$

If we have 7 or more claims during the first day, then the second condition is automatically satisfied.

If we have exactly 6 claims on the first day, then we need at least 1 claim on the second day.

If we have exactly 5 claims on the first day, then we need at least 2 claims on the second day.

So the required probability is:

$P(N_G(1) >=7) + P(N_G(1) =6, N_G(2)- N_G(1)>=1)$
$+ P(N_G(1) =5, N_G(2)- N_G(1)>=2)$ ---------------------------------(1)

Now $N_G(1)$ and $N_G(2)- N_G(1)$ are both independent Poisson with mean 0.3*4 = 1.2 .

$P(N_G(1) >=7) = 1- (N_G(1) <=6) = 1- \exp(-1.2)* (1+ 1.2/1!+1.2^2/2! + 1.2^3/3!+ 1.2^4/4! + 1.2^5/5!+ 1.2^6/6!)$
=1- .999749
=0.000251

$P(N_G(1) =6, N_G(2)- N_G(1)>=1) = P(N_G(1) =6)* P(N_G(2)- N_G(1))>=1)$

$= P(N_G(1) =6)* \{ 1- P(N_G(2)- N_G(1))= 0)$
$= \{\exp(-1.2)* 1.2^6/6! \} * \{1- \exp(-1.2)\}$
$= 0.000873$

$P(N_G(1) =5, N_G(2)- N_G(1)>=2) = P(N_G(1) =5)* \{ 1- P(N_G(2)- N_G(1)) <=1\}$
$= \{\exp(-1.2)* 1.2^5/5! \} *$
$\{1- \exp(-1.2)(1+1.2)\}$
=0.006246*0.337373
=0.002107

Substituting in (1),

The required probability = 0.000251+ 0.000873+0.002107
= 0.003231