PUSASQF505A
Statistical and Risk Modelling - 3

Time: 2 hours
Total Marks: 60 marks

Note:

1. **The candidate has option to either attempt question 4A or question 4B. Rest all questions are mandatory.**
2. **Numbers to the right indicate full marks.**
3. **The candidates will be provided with the formula sheet and graph papers (if required) for the examination.**
4. **Use of approved scientific calculator is allowed.**

Q1 A **5 Marks**

(a) Define what is meant by a Poisson Process. (2)

ANSWER:
A Poisson process is a model for a series of discrete events where the average time between events is known, but the exact timing of events is random.

The city of Mumbai is known for its Local train network. The train route from Ghatkopar has one train scheduled every 15 minutes. Rail conditions in Mumbai are such that arrival times of trains at any station can be assumed to follow a Poisson process. The trains arriving could be a Fast train or a slow train. The probability that any train is a Fast train is 30%.
Mr Manav arrives at the station at 9am to find that there is no train at the station. He wishes to get on the first Fast train to arrive.

(b) Calculate the probability that the first train that arrives at Ghatkopar after 9am will be a Slow train. (1)

ANSWER:

$$P(Slow\ Train) = 1 - P(Fast\ Train) = 1 - 0.3 = 0.7$$

(c) Determine the probability that Mr. Manav will not be able to catch a train by 9:09am. (2)

$$Fast\ Train \sim Poi\left(\frac{1}{15} * 0.3\right) per\ minute$$

$$Fast\ Train \sim Poi\left(\frac{1}{50}\right) per\ minute$$

$$P(No\ Fast\ Trains\ in\ 9\ mins) = e^{-\frac{9}{50}} = 0.83527$$

Q1 B **5 Marks**

A life insurance company is currently conducting the analysis of its portfolio and trying to model policies in the following states , Active (A), Surrendered (S) , Paid Up (P) and Claimed (C). They are using a time inhomogeneous Markov Jump process {X(t) : t>=0)

The transition rates at age t are shown below:

	A	S	P	C
A	-0.65t	0.1t	0.5t	0.05t
S	0	-0.01t	0	0.01t

P	0.2t	0.15t	-0.4t	0.05t
C	0	0	0	0

(a) Start from the Chapman-Kolmogorov equation for $P_{ij}(s, t)$ and differentiate it to obtain the backward differential equation for probability $P_{PS}(s, t)$

(5)

ANSWER:

$$p_{ij}(s, t + h) = \sum_{k\in S} p_{ik}(s, t)p_{kj}(t, t + h)$$

For small h, we know that

$$p_{kj}(t, t + h) = h\mu_{kj}(t) + o(h),\ for\ j\neq k$$

$$p_{jj}(t, t + h) = 1 + h\mu_{jj}(t) + o(h)$$

So

$$p_{ij}(s, t + h) = \sum_{k\neq j} p_{kj}(s, t)h\mu_{kj}(t) + p_{ij}(s, t)\left(1 + h\mu_{jj}(t)\right) + o(h)$$

$$p_{ij}(s, t + h) = p_{ij}(s, t) + \sum_{k\in S} p_{ik}(s, t)h\mu_{kj}(t) + o(h)$$

$$\frac{p_{ij}(s,t+h)-p_{ij}(s,t)}{h} = \sum_{k\in S} p_{ik}(s, t)\mu_{kj}(t) + \frac{o(h)}{h}$$

And letting h tends to 0, we get

$$\frac{\partial}{\partial t}p_{ij}(s, t) = \sum_{k\in S} p_{ik}(s, t)\mu_{kj}(t)$$

Candidates need to further substitute the required state space values for i and j

Q1 C **5 Marks**

An investigation was carried out into the relationship between morbidity and mortality in an historical population of actuarial analysts. The investigation used a three state model with the states"

H : Healthy
S : Sick
D : Dead

Let the probability that a person in state *i* at time t will be in state j at time t+s be ${}_{s}p_{t}^{ij}$.

Let the transition intensity at time s + t between any two states i and j by μ_{s+t}^{ij}

(a) Show from first principles that:

$$\frac{\partial}{\partial s}\,{}_{s}p_{t}^{SD} = {}_{s}p_{t}^{SH}\mu_{s+t}^{HD} + {}_{s}p_{t}^{SS}\mu_{s+t}^{SD}$$

ANSWER:
Please refer the derivation in Q1B.

Q2 A **5 Marks**

A game show has recently gained public attention in India. The game show "Kya aap panchvi pass se tez hai?" has a very simple concept. The contestants are asked a question and they have to answer quicker than a 5^{th} grade student selected by the host. Rahul had recently participated in the game show and this was the outcome of his first 18 questions.

LLLWWWLLWDDWLWWDLD

Where L represent that Rahul was slowed than 5^{th} grader, W represents that Rahul was faster than the 5^{th} grader and D represents that both required equal time to answer the question.

Assume that the probability of the result for the next question depends only on the result of the current question and no prior question.

(a) Estimate the transition matrix for the Markov Chain.

(3)

(b) Calculate the probability that the Rahul takes lesser time on the 20^{th} question.

(2)

ANSWER:

	D	L	W
D	1/3	1/3	1/3
L	0.1429	0.4286	0.4286
W	0.2857	0.2857	0.4286

Based on the two step transition matrix,
P^2[D,W] = 0.4013605

Q2 B **5 Marks**

A large pensions agency operates a employee benefit scheme which pays a sickness benefit to any employee unable to work due to ill health and a death benefit to any employee who dies while in service.

(a) Give the likelihood of the data, defining all the terms you use.

(2)

ANSWER:

$$L = e^{-(\mu_{HS}+\mu_{HD})(t_H)}\mu_{HS}^{n_{HS}}\mu_{HD}^{n_{HD}} + e^{-(\mu_{SH}+\mu_{SD})*t_S}\mu_{SH}^{n_{SH}}\mu_{SD}^{n_{SD}}$$

$$\mu_{ij}\, is\ the\ rate\ of\ trnasition\ from\ state\ i\ to\ j$$

$$n_{ij}\ is\ the\ number\ of\ transitions\ observed\ from\ state\ i\ to\ j$$

$$t_i\, is\ the\ total\ time\ spend\ in\ state\ i$$

(b) Derive the maximum likelihood estimate of the rate of falling sick.

(3)

ANSWER:

$$l = constant - \left(\mu_{HS} + \mu_{HD}\right)t_H + n_{HS}\ln ln\,\mu_{HS}$$

$$\frac{dl}{d\mu_{HS}} = \ -t_H + \frac{n_{HS}}{\mu_{HS}}$$

$$Therefore$$

$$\hat{\mu}_{HS} = \frac{n_{HS}}{t_H}$$

Q2 C **5 Marks**

The child at a particular day care center is classified as being Silent (S) or Noisy (N). The status is assumed to follow a Markov jump process with time homogeneous transition rates μ_{SN} and μ_{NS}

(a) Write down the Kolmogorov's forward equation for $\frac{\partial}{\partial t}\,{}_tp_s^{\overline{SS}}$ and $\frac{\partial}{\partial t}\,{}_tp_s^{NS}$

(2)

ANSWER:

$$\frac{\partial}{\partial t}\,{}_tp_s^{\overline{SS}} = -\,{}_tp_s^{\overline{SS}}\,\mu_{SN}$$
$$\frac{\partial}{\partial t}\,{}_tp_s^{NS} = -\,{}_tp_s^{NS}\,\mu_{SN} + {}_tp_s^{NN}\,\mu_{NS}$$

Viyansh is noisy when he enters the class at 9am in the morning.

(b) Using your answer in (a), derive the expression for the probability that he will be silent by recess time at 12 noon.

(3)

ANSWER:

Q3 A **5 Marks**

A continuous time Markov model for Actuarial Exams has four states: CM1 , CS1 ,CS2 and CM2

From CM1, the transitions are possible to states CS1 and CM2 at 0.2 and 0.05 per year respectively. From CS1 , a person transitions to CM1 at a rate of 1 per year, to CS2 with 0.5 and to CM2 at a rate of 0.02 per year. Only one transition is possible from the CS2 state to CM2 at a rate of 0.4 per year. CM2 is an absorbing state.

(a) Let T_j denote the expected time in which a life which is currently in state j will reach State CM2.
Calculate the values of T_j for all states j

ANSWER:

$$T_{CM2} = 0$$
$$T_{CS2} = \frac{1}{0.4} + T_{CM2} = 2.5 + 0 = 2.5$$
$$T_{CS1} = \frac{1}{1+0.5} + \frac{1}{1.5} * T_{CM1} + \frac{0.5}{1.5} * T_{CS2} = \frac{2}{3} + \frac{2}{3}T_{CM1} + \frac{2.5}{3}$$
$$T_{CM1} = \frac{1}{0.2+0.05} + \frac{4}{5}T_{CS1} + \frac{1}{5}T_{CM1} = 4 + \frac{4}{5}T_{CS1}$$
$$T_{CS1} = \frac{2}{3} + \frac{8}{3} + \frac{8}{15}T_{CS1} + \frac{2.5}{3}$$
$$T_{CS1} = \frac{12.5}{3} * \frac{15}{7} = \frac{62.5}{7}$$
$$T_{CM1} = 11.1428$$

Q3 B **5 Marks**

Flipzon has recently updated its work policy to provide flexibility to its employees but also reduce the chances of laptops getting water damage due to rains. Flipzon provides all employees with two laptops. If it is raining at the time the employees leaves (from office or from home) , they leave the laptop and travel without the laptop. If its not raining at the time they leave and atleast one laptop is available where they are, they carry one laptop with them to the other place. i.e. if its not raining, the employee carries the laptop from home to office.

The probability of it raining on any particular journey is p.

The number of laptops is examined as a Markov Chain with state space (0,1,2) representing the number of umbrellas at the employee's current location (office or home) and each time step representing one journey.

(a) Derive the transition matrix for the number of laptops at the office before he leaves each evening, based on the number of laptops he leaves the previous evening.

ANSWER:

The one step transition matrix P is:

	0	1	2
0	0	0	1
1	0	P	1-p
2	p	1-p	0

Since the above one gives number of laptops at home given the number of laptops in office. We need number of laptops at office given the number of laptops at office the previous evening, we need to use the square of this matrix:

P^2 *is given by*:

	0	1	2
0	p	1-p	0
1	p(1-p)	p^2+(1-p)^2	p(1-p)
2	0	p(1-p)	p+(1-p)^2

Q3 C **5 Marks**

For a Markov Chain with 6 states, the transition matrix P, as is given below:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

(a) Find the steady state distribution for the above Markov Chain

$$\pi = c\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

Q4 A **15 Marks**

An extremely lavish township is being built at an hill station. In order to attract the extremely net worth individuals the builder is promoting an private jet hanger (i.e. parking for planes). A prospective purchaser of land in the township is concerned that he would return in his jet only to find that there was no empty hanger to park the jet. He asks you to model the number of occupied hangers at any time using a time homogeneous Markov jump process.

- There are 4 airplane hangers at the township.
- The rate at which an airplane arrives to seek an hanger is μ
- For each airplane which is currently parked, the probability that its owner takes the plane away in a short interval h is $\lambda h + o(h)$
  Where $\lambda > 0$ and $\mu > 0$

(a) Draw a transition diagram as well as a generator matrix for this process. Please specify what each state represents.

(4)

	0	1	2	3	4
0	$-\mu$	μ	0	0	0
1	λ	$-\lambda-\mu$	μ	0	0
2	0	2λ	$-2\lambda-\mu$	μ	0
3	0	0	3λ	$-3\lambda-\mu$	μ
4	0	0	0	4λ	-4λ

0 represents empty and 4 represents full

(b) Derive the probability that, given that hangers are full, they will remain full for at least the next 6 hours.

(2)

ANSWER:

$$p_{\overline{44}}\left(\frac{1}{4}\right) = e^{-\lambda}$$

$$\textit{Assuming that } \lambda \textit{ is rate in days}$$

(c) Specify the transition matrix for the jump chain associated with this process.

(2)

	0	1	2	3	4
0	0	1	0	0	0
1	$\frac{\lambda}{\lambda+\mu}$	0	$\frac{\mu}{\lambda+\mu}$	0	0
2	0	$\frac{2\lambda}{2\lambda+\mu}$	0	$\frac{\mu}{2\lambda+\mu}$	0
3	0	0	$\frac{3\lambda}{3\lambda+\mu}$	0	$\frac{\mu}{3\lambda+\mu}$
4	0	0	0	1	0

Suppose only two of the four hangers are currently full

(d) Determine the probability that all the hangers become full before any planes leave.

(2)

ANSWER:

$$Let\ p\ be\ the\ probability$$

$$p = P_{23} * P_{34} = \frac{\mu}{2\lambda+\mu}\frac{\mu}{3\lambda+\mu} = \frac{\mu^2}{(2\lambda+\mu)(3\lambda+\mu)}$$

(e) Derive the probability that the hangers become full before the hanger becomes fully empty.

(3)

$$Let\ p\ be\ the\ probability$$

$$p = p_{23}p_{34} + p_{21}p_{12}p_{23}p_{34} + \left(p_{21}p_{12}\right)^2 p_{23}p_{34} + \ ....$$

$$p = \frac{p_{23}p_{34}}{1-p_{21}p_{12}}$$

(f) Comment on the purchaser's assumptions regarding the arrival and departure of airplanes

(2)

ANSWER:
Accept any reasonable comments.

OR

Q4 B

Club 6E is the newest frequent flyer program in India. It offers additional perks and discounts to the most loyal customers of 6E Air. The loyalty program has four tiers , Base (B), Silver (S), Gold (G) and Platinum (P). The Silver , Gold and Platinum customers get 10% , 25% and 50% respectively.
The rules of the program are as follows:
Members who book two or more flights in a given calender year move up one class for the following year (or remain Platinum members)
Members who book no flights in a given calender year move down one class or remain Base members.
Members need to take one flight to maintain their loyalty status.
Let the proportions of members booking 0,1 and 2+ flights in a given year be p_0 , p_1 and p_2 respectively.

(a) Explain what is meant by a Markov Chain and how this chain can be modelled as a Markov Chain.

(3)

ANSWER:
The process can be modelled as a Markov chain with four states corresponding to the four classes. B S G and P.
The process has discrete states and operates in discrete time, with movements occurring only at the end of each calendar year. This set-up has the Markov property because all the members in a particular state in a particular year have the same probabilities for moving between the states in the future.

(b) Explain the conditions under which a Markov Chain has a unique stationary distribution. Also explain how those conditions are met in this case.

(2)

ANSWER:

A stationary distribution is a set of probabilities π_i that satisfy the matrix equations:

$$\pi = \pi P\ ,\ \sum_i \pi_i = 1\ ,\ \pi_i \geq 0$$

This chain has a finite state space, with 4 states. So there is at least one stationary distribution.

The chain is also irreducible and hence it has one and only one stantionary distribution.

6E analysts have provided you with the information that 40% of members book no flights and 20% members book more than two flights in any given year.

(c) Calculate the unique stationary distribution.

(5)

ANSWER:

We are told that $p_0 = 0.4$, $p_1 = 0.4$ and $p_{2+} = 0.2$. So the transition matrix for the model is:

$$\begin{array}{c} \\ O \\ B \\ S \\ G \end{array} \begin{array}{c} \begin{array}{cccc} O & B & S & G \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.4 & 0.4 & 0.2 & 0 \\ 0 & 0.4 & 0.4 & 0.2 \\ 0 & 0 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

The stationary distribution is the set of probabilities π_i that satisfy the matrix equation $\pi = \pi P$ with the additional condition $\sum_i \pi_i = 1$.

Written out in full, the matrix equation for the stationary distribution $(\pi_O, \pi_B, \pi_S, \pi_G)$ satisfies the system of equations:

$$\pi_O = 0.8\pi_O + 0.4\pi_B \qquad (1)$$
$$\pi_B = 0.2\pi_O + 0.4\pi_B + 0.4\pi_S \qquad (2)$$
$$\pi_S = 0.2\pi_B + 0.4\pi_S + 0.4\pi_G \qquad (3)$$
$$\pi_G = 0.2\pi_S + 0.6\pi_G \qquad (4)$$

From Equation (1), we have:

$$0.2\pi_O = 0.4\pi_B \qquad \Rightarrow \pi_B = 0.5\pi_O$$

Equation (2) then becomes:

$$0.5\pi_O = 0.2\pi_O + 0.4(0.5\pi_O) + 0.4\pi_S$$

So:

$$0.1\pi_O = 0.4\pi_S \qquad \Rightarrow \pi_S = 0.25\pi_O$$

(State named used is O,B,S,G instead of B,S,G,P)
Solving the equations:

$$\pi = \left(\frac{8}{15}, \frac{4}{15}, \frac{2}{15}, \frac{1}{15}\right)$$

6E executives have asked the team to ascertain if the frequent flyer program will be profitable.
For their analysis, they have assumed that cost of providing the benefit to the Base customer is Rs. 0, Silver customer is Rs. 1000, Gold customer is Rs. 2000 and Platinum customer is Rs.5000
The airline makes a profit of Rs. 1200 per passenger for every flight before taking into account the cost of managing a frequent flyer program

(d) Assess and comment on whether the frequent flyer program makes the airline profits.

(5)

ANSWER:

Cost = $0 * \pi_B + 1000 * \pi_S + 2000 * \pi_G + 5000 * \pi_P = 866.67$

Profit = $0 * p_0 + 1200 * p_1 + 1200 * n * p_{2+}$

Whwere n is the average number of flights booked by members booking 2 or more.

So to make an overall profit, we need

$$1200 * 0.4 + 1200 * n * 0.2 > 866.67$$
$$480 + 240 * n > 867.67$$
$$n > 1.6111$$

Since n must be least 2, we can conclude that the scheme is profitable.