INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical and risk Modelling 3

Chapter: Unit 2

Category: Practice Questions

1. CT4 April 2010 Q4

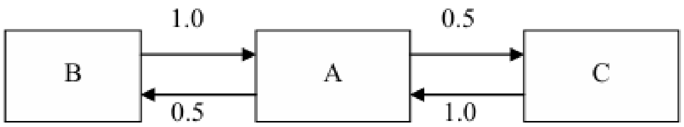
A Markov Chain with space {A, B, C} has the following properties:

- It is irreducible
- It is periodic
- The probability of moving from A to B equals the probability moving from A to C.

(i) Show that these properties uniquely define the process.

(ii) Sketch a transition diagram for the process.

Answer:

[(ii) 

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2. CT4 April 2011 Q4

Children at a school are given weekly grade sheets, in which their effort is graded in four levels : 1”Poor”, 2 “Satisfactory”, 3”Good” and 4 “Excellent”. Subject to a maximum level of Excellent and a minimum level of Poor, between each week and the next, a child has :

- a 20 per cent chance of moving up one level
- a 20 per cent chance of moving down one level
- a 10 per cent chance of moving up two level
- a 10 per cent chance of moving down two level

moving up or down three levels in a single week is not possible.

(i) Write down the transition matrix of his process.

Children are graded on Friday afternoon in each week. On Friday of the first week of the school year, as there is little evidence on which to base an assessment, all children are graded “Satisfactory”.

(ii) Calculate the probability distribution of the process after the grading on Friday of the third week of the school year.

Answer:

[(i) $\begin{pmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0.1 & 0.2 & 0.7 \end{pmatrix}$, (ii) 35% that a child will be graded Poor', 27% that a child will be graded Satisfactory, 21% that a child will be graded Good and 17% that a child will be graded Excellent.]

3. CT4 April 2010 Q10

An airline runs a frequent flyer scheme with four classes of member: in ascending order Ordinary, Bronze, Silver and Gold. Members receive benefits according to their class. Members who book two or more flights in a given calendar year move up one class for the following year (or remain Gold members), members who book exactly one flight in a given calendar year stay at the same class, and members who book no flights in a given calendar year move down one class (or remain Ordinary members).

Let the proportions of members booking 0, 1 and 2+ flights in a given year be p0, p1 and p2+ respectively.

(i) (a) Explain how this scheme can be modelled as a Markov chain.

(b) Explain why there must be a unique stationary distribution for the proportion of members in each class.

(ii) Write down the transition matrix of the process.

The airline's research has shown that in any given year, 40% of members book no flights, 40% book exactly one flight, and 20% book two or more flights.

(iii) Calculate the stationary probability distribution.

The cost of running the scheme per member per year is as follows:

- Ordinary members £0
- Bronze members £10
- Silver members £20
- Gold members £30

The airline makes a profit of £10 per passenger for every flight before taking into account costs associated with the frequent flyer scheme.

(iv) Assess whether the airline makes a profit on the members of the scheme.

Answer:

The transition matrix P is:

[(iii) $$\begin{array}{c} O \\ B \\ S \\ G \end{array}\begin{pmatrix} p_0+p_1 & p_{2+} & 0 & 0 \\ p_0 & p_1 & p_{2+} & 0 \\ 0 & p_0 & p_1 & p_{2+} \\ 0 & 0 & p_0 & p_1+p_{2+} \end{pmatrix},$$

(iv) The stationary distribution is, $\pi_O=\dfrac{8}{15}=0.5333$, $\pi_B=\dfrac{4}{15}=0.2667$, $\pi_S=\dfrac{2}{15}=0.1333$, $\pi_G=\dfrac{1}{15}=0.0667$]

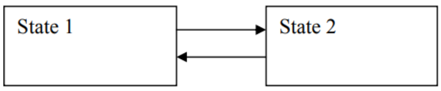
4. CT4 October 2011 Q1

The diagram below show three Markov chains, where arrows indicate a non- zero transition probability:

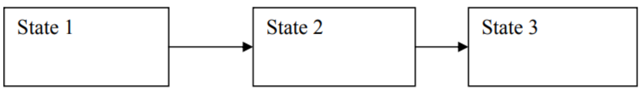
State whether each of the chain is:

(a) Irreducible
(b) Periodic, giving the period where relevant.

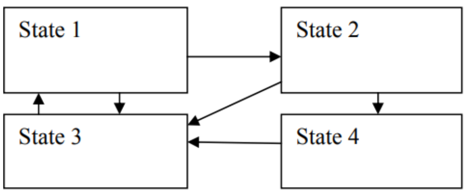
A.



B.



C.



Answer:

[(a) A- Yes, B – No, C- Yes, (ii) A – Yes , B- No, C – No]

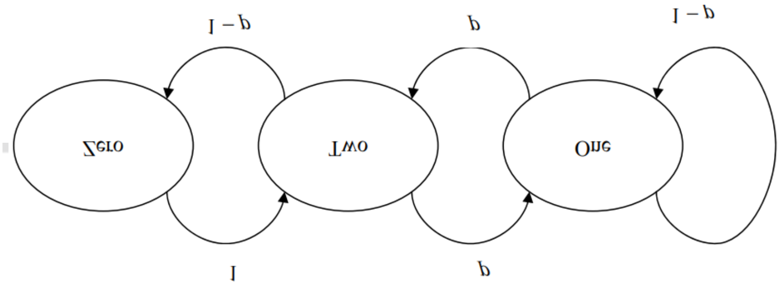
5. CT4 September 2011 Q11

An actuary walks from his house to the office each morning, and walks back again each evening. He owns two umbrellas. If it is raining at the time he sets off, and one or both of his umbrellas is available, he takes an umbrella with him. However if it is not raining at the time he sets off he always forgets to take an umbrella.

Assume that the probability of it raining when he sets off on any particular journey is a constant p, independent of other journeys.

This situation is examined as a Markov Chain with state space {0,1,2} representing the number of his umbrellas at the actuary's current location (office or home) and each time step representing one journey.

(i) Explain why the transition graph for this process is given by:



(ii) Derive the transition matrix for the number of umbrellas at the actuary's house before he leaves each morning, based on the number before he leaves the previous morning.

(iii) Calculate the stationary distribution for the Markov Chain.

(iv) Calculate the long run proportion of journeys (to or from the office) on which the actuary sets out in the rain without an umbrella.

The actuary considers that the weather at the start of a journey, rather than being independent of past history, depends upon the weather at the start of the previous journey. He believes that if it was raining at the start of a journey the probability of it raining at the start of the next journey is r ($0 < r < 1$,) and if it was not raining at the start of a journey the probability of it raining at the start of the next journey is s ($0 < s < 1$, $r \neq s$).

(v) Write down the transition matrix for the Markov Chain for the weather.

(vi) Explain why the process with three states {0,1,2}, being the number of his umbrellas at the actuary's current location, would no longer satisfy the Markov property.

(vii) Describe the additional state(s) needed for the Markov property to be satisfied and draw a transition diagram for the expanded system.

Answer:

One step transition matrix is:

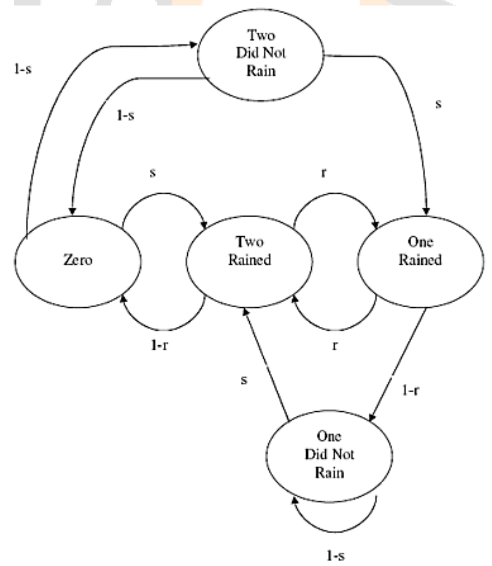
[(ii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{pmatrix}$, Two-step transition matrix - $\begin{pmatrix} 1-p & p & 0 \\ p(1-p) & (1-p)^2+p^2 & p(1-p) \\ 0 & p(1-p) & 1-p+p^2 \end{pmatrix}$,

(iii) $\pi_1 = \dfrac{1-p}{3-p}$, $\pi_2 = \pi_3 = \dfrac{1}{3-p}$, (iv) $p.\pi_1 = \dfrac{p(1-p)}{3-p}$, (v)

$From/To$	R	NR
R	r	$1-r$
NR	s	$1-s$

,

(vii) The state space required is {Zero, One Rained, One Did Not Rain, Two Rained, Two Did Not Rain}



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6. CT4 September 2012 Q5

A no claims discount system operates with three levels of discount, 0%, 15% and 40%. If a policyholder makes no claim during the year he moves up a level of discount (or remains at the maximum level). If he makes one claim during the year he moves down one level of discount (or remains at the minimum level) and if he makes two or more claims he moves down to, or remains at, the minimum level.

The probability for each policyholder of making two or more claims in a year is 25% of the probability of making only one claim.

The long-term probability of being at the 15% level is the same as the long-term probability of being at the 40% level.

(i) Derive the probability of a policyholder making only one claim in a given year.

(ii) Determine the probability that a policyholder at the 0% level this year will be at the 40% level after three years.

(iii) Estimate the probability that a policyholder at the 0% level this year will be at the 40% level after 20 years, without calculating the associated transition matrix.

Answer:

[(i) 0.4, (ii) 0.25, (iii) 0.3125]

7. CT4 April 2013 Q11

(i) Explain what is meant by a time inhomogeneous Markov chain and give an example of one.

A No Claims Discount system is operated by a car insurer. There are four levels of discount: 0%, 10%, 25% and 40%. After a claim-free year a policy holder moves up one level (or remains at the 40% level). If a policy holder makes one claim in a year he or she moves down one level (or remains at the 0% level). A policy holder who makes more than one claim in a year moves down two levels (or moves to or remains at the 0% level). Changes in level can only happen at the end of each year.

(ii) Describe, giving an example, the nature of the boundaries of this process.

(iii) (a) State how many states are required to model this as a Markov chain.

(b) Draw the transition graph.

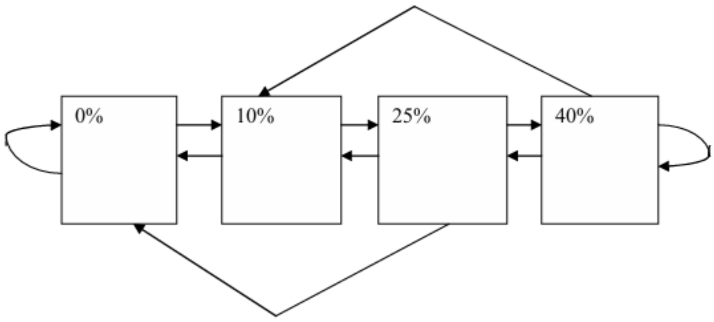
The probability of a claim in any given month is assumed to be constant at 0.04. At most one claim can be made per month and claims are independent.

(iv) Calculate the proportion of policyholders in the long run who are at the 25% level.

(v) Discuss the appropriateness of the model.

Answer:

Four states are required: 0%, 10%, 25% and 40%.



[(iii) , (iv) 24.47%]

8. CT4 September 2013 Q2

The two football teams in a particular city are called United and City and there is intense rivalry between them. A researcher has collected the following history on the results of the last 20 matches between the teams from the earliest to the most recent, where:

U indicates a win for United;

C indicates a win for City;

D indicates a draw.

UCCDDUCDCUUDUDCCUDCC

The researcher has assumed that the probability of each result for the next match depends only on the most recent result. He therefore decides to fit a Markov chain to this data.

(i) Estimate the transition probabilities for the Markov chain.

(ii) Estimate the probability that United will win at least two of the next three matches against City.

Answer:

[(i)

From/To	U	C	D
U	1/6	1/3	1/2
C	2/7	3/7	2/7
D	1/3	1/2	1/6

, (ii) 0.15873]

9. CT4 September 2013 Q5

A motor insurer offers a No Claims Discount scheme which operates as follows. The discount levels are {0%, 25%, 50%, 60%}. Following a claim-free year a policyholder moves up one discount level (or stays at the maximum discount). After a year with one or more claims the policyholder moves down two discount levels (or moves to, or stays in, the 0% discount level).

The probability of making at least one claim in any year is 0.2.

(i) Write down the transition matrix of the Markov chain with state space {0%, 25%, 50%, 60%}.

(ii) State, giving reasons, whether the process is:

(a) irreducible.

(b) aperiodic.

(iii) Calculate the proportion of drivers in each discount level in the stationary distribution.

The insurer introduces a “protected” No Claims Discount scheme, such that if the 60% discount is reached the driver remains at that level regardless of how many claims they subsequently make.

(iv) Explain, without doing any further calculations, how the answers to parts (ii) and (iii) would change as a result of introducing the “protected” No Claims Discount scheme.

Answer:

[(i) $P = \begin{pmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.2 & 0 & 0.8 & 0 \\ 0.2 & 0 & 0 & 0.8 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix}$, (ii)a- irreducible, b- aperiodic, (iii) 0% = 0.08257, 25% = 0.18349, 50% = 0.14679, 60% = 0.58716]

10. CT4 April 2014 Q10

An industrial kiln is used to produce batches of tiles and is run with a standard firing cycle. After each firing cycle is finished, a maintenance inspection is undertaken on the heating element which rates it as being in Excellent, Good or Poor condition, or notes that the element has Failed.

The probabilities of the heating element being in each condition at the end of a cycle, based on the condition at the start of the cycle are as follows:

START	*END* *Excellent*	*Good*	*Poor*	*Failed*
Excellent	0.5	0.2	0.2	0.1
Good		0.5	0.3	0.2
Poor			0.5	0.5
Failed				1

(i) Write down the name of the stochastic process which describes the condition of a single heating element over time.

(ii) Explain whether the process describing the condition of a single heating element is:
(a) irreducible.
(b) periodic.

(iii) Derive the probability that the condition of a single heating element is assessed as being in Poor condition at the inspection after two cycles, if the heating element is currently in Excellent condition.

If the heating element fails during the firing cycle, the entire batch of tiles in the kiln is wasted at a cost of £1,000. Additionally, a new heating element needs to be installed at a cost of £50 which will, of course, be in Excellent condition.

(iv) Write down the transition matrix for the condition of the heating element in the kiln at the start of each cycle, allowing for replacement of failed heating elements.

(v) Calculate the long-term probabilities for the condition of the heating element in the kiln at the start of a cycle.

The kiln is fired 100 times per year.

(vi) Calculate the expected annual cost incurred due to failures of heating elements.

The company is concerned about the cost of ruined tiles and decides to change its policy to replace the heating element if it is rated as in Poor condition.

(vii) Evaluate the impact of the change in replacement policy on the profitability of the company.

Answer:
[(i) Markov chain, (ii) a - not irreducible, b - not periodic, (iii) 0.26.,

(iv)

	Excellent	*Good*	*Poor*
Excellent	0.6	0.2	0.2
Good	0.2	0.5	0.3
Poor	0.5	0	0.5

, (v) $\pi_E = \frac{25}{51}, \pi_G = \frac{10}{51}, \pi_P = \frac{16}{51}$, (vi) £25,735,

(vii) Overall profits improved by £11,092]

11. CT4 September 2014 Q5

A sports league has two divisions {1,2} with Division 1 being the higher. Each season the bottom team in Division 1 is relegated to Division 2, and the top team in Division 2 is promoted to Division 1.

Analysis of the movements of teams between divisions indicates that the probabilities of finishing top or bottom of a division differs if a team has just been promoted or relegated, compared with the probabilities in subsequent seasons.

The probabilities are as follows:

Finishing position	*If promoted previous season*	*If relegated previous season*	*If neither promoted nor relegated previous season*
Top	0.1	0.25	0.15
Bottom	0.3	0.25	0.15
Other	0.6	0.5	0.7

(i) Write down the minimum number of states required to model this as a Markov chain.

(ii) Draw a transition graph for the Markov chain.

(iii) Write down the transition matrix for the Markov chain.

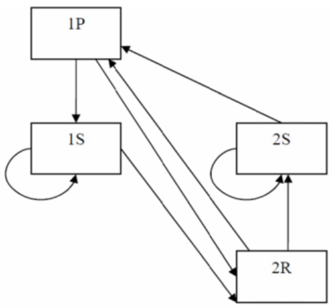
(iv) Explain whether the Markov chain is:
(a) irreducible.
(b) aperiodic.

Team A has just been promoted to Division 1.

(v) Calculate the minimum number of seasons before there is at least a 60% probability of Team A having been relegated to Division 2.

Answer:

[(i) 4, (ii)



, (iii)

	1P	1S	2S	2R
1P	0	0.7	0	0.3
1S	0	0.85	0	0.15
2S	0.15	0	0.85	0
2R	0.25	0	0.75	0

, (iv) a – Yes, b- yes,
(v) Year 5]

12. CT4 September 2014 Q6

A motor insurance company offers annually renewable policies. To encourage policyholders to renew each year it offers a No Claims Discount system which reduces the premiums for those people who claim less often. There are four levels of premium:

0: no discount
1: 15% discount
2: 25% discount
3: 40% discount

A policyholder who does not make a claim in the year, moves up one level of discount the following year (or stays at the maximum level).

A policyholder who makes one or more claims in a year moves down one level of discount if they did not claim in the previous year (or remains at the lowest level) but if they made at least one claim in the previous year they move down two levels of discount (subject to not going below the lowest level).

(i) (a) Explain how many states are required to model this as a Markov chain.
(b) Draw the transition graph of the process.

The probability, p, of making at least one claim in any year is constant and independent of whether a claim was made in the previous year.

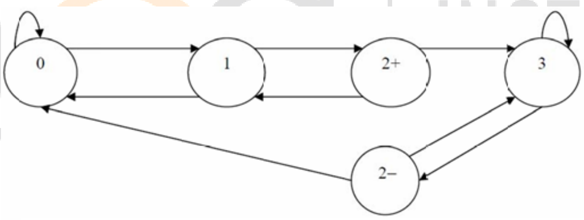
(ii) Calculate the proportion of policyholders who are at the 25% discount level in the long run given that the proportion at the 40% level is nine times that at the 15% level.

(iii) (a) Explain how the state space of the process would change if the probability of making a claim in any one year depended upon whether a claim was made in the previous year.

(b) Write down the transition matrix for this new process.

Answer :

[(i) a- 5, b



, (ii) 0.213018, (iii) a- 6 states,

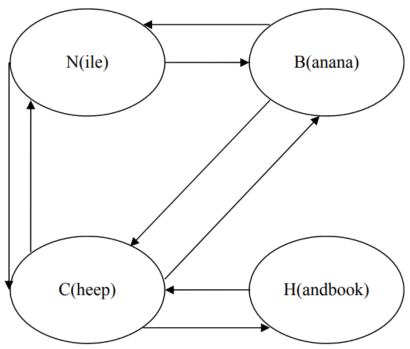
$$\begin{pmatrix} r & 1-r & 0 & 0 & 0 & 0 \\ n & 0 & 0 & 1-n & 0 & 0 \\ r & 0 & 0 & 1-r & 0 & 0 \\ 0 & 0 & n & 0 & 0 & 1-n \\ r & 0 & 0 & 0 & 0 & 1-r \\ 0 & 0 & 0 & 0 & n & 1-n \end{pmatrix}$$

b - where the levels are ordered $0, 1^+, 1^-, 2^+, 2^-, 3$.]

13. CT4 April 2015 Q7

(i) Describe what is meant by a Markov chain.

A simplified model of the internet consists of the following websites with links between the websites as shown in the diagram below:



An internet user is assumed to browse by randomly clicking any of the links on the website he is on with equal probability.

(ii) Calculate the transition matrix for the Markov chain representing which website the internet user is on.

(iii) Calculate, of the total number of visits, what proportion are made to each website in the long term.

Answer:

[(ii) $$P = \begin{array}{ccccc} From\backslash To & N & B & C & H \\ N & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ B & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ C & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ H & 0 & 0 & 1 & 0 \end{array}$$, (iii) $\pi_N = \frac{1}{4}, \pi_B = \frac{1}{4}, \pi_C = \frac{3}{8}, \pi_H = \frac{1}{8}$]

14. CT4 September 2015 Q10

A profession has examination papers in two subjects, A and B, each of which is marked by a team of examiners. After each examination session, examiners are given the choice of remaining on the same team, switching to the other team, or taking a session's holiday.

In recent sessions, 10% of subject A's examiners have elected to switch to subject B and 10% to take a holiday. Subject B is more onerous to mark than subject A, and in recent sessions, 20% of subject B's examiners have elected to take a holiday in the next session, with 20% moving to subject A.

After a session's holiday, the profession allocates examiners equally between subjects A and B. No examiner is permitted to take holiday for two consecutive sessions.

(i) Sketch the transition graph for the process.

(ii) Determine the transition matrix for this process.

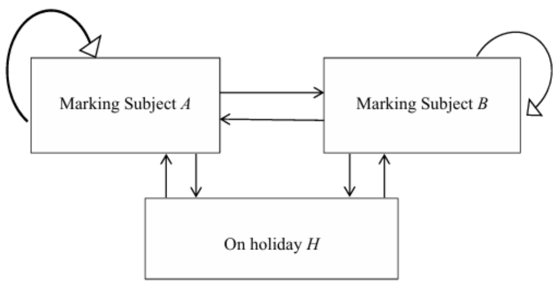
(iii) Calculate the proportion of the profession's examiners marking for subjects A and B in the long run.

The profession considers that in future, an equal number of examiners is likely to be required for each subject. It proposes to try to ensure this by adjusting the proportion of those examiners on holiday who, when they return to marking, are allocated to subjects A and B.

(iv) Calculate the proportion of examiners who, on returning from holiday, should be allocated to subject B in order to have an equal number of examiners on each subject in the long run.

Answer:

[(i)



, (ii)

$$\begin{array}{r} \text{Subject A} \\ \text{Subject B} \\ \text{Holiday} \end{array} \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

, (iii) A - 58.8% ,29.4% - B, (iv) All]

15. CT4 April 2016 Q10

In a small country there are only four authorised insurance companies, A, B, C and D. The law in this country requires homeowners to take out buildings insurance from an authorised insurance company. All policies provide cover for a period of one year.

Based on analysis of the compliance database used to check that every home is insured, the probabilities of a homeowner transferring between the four companies at the end of each year are considered to be described by the following transition matrix:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{pmatrix} 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.4 & 0.1 \\ 0 & 0.2 & 0.2 & 0.6 \end{pmatrix}$$

Yolanda has just bought her policy from Company C for the first time.

(i) Calculate the probability that Yolanda will be covered by Company C for at least five years before she changes provider.

Zachary took out a policy with Company A in January 2013. Unfortunately, Zachary's house burnt down on 12 March 2015.

(ii) Calculate the probability that Company A does NOT cover Zachary's home at the time of the fire.

(iii) Calculate the expected proportions of homeowners who are covered by each insurance company in the long run.

Company A makes an offer to buy Company D. It bases its purchase price on the assumption that homeowners who would previously have purchased policies from Company A or Company D would now buy from the combined company, to be called Addda.

(iv) Determine the transition matrix which will apply after the takeover if Company A's assumption about homeowners' behaviour is correct.

(v) Comment on the appropriateness of Company A's assumption.

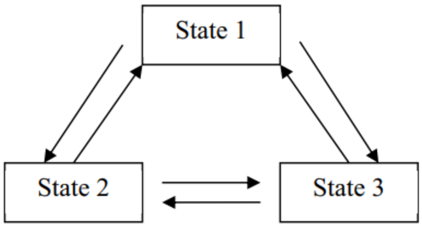
Answer:

[(i) 0.0256, (ii) 0.65, (iii) $\pi_A = 31/120, \pi_B = 1/3, \pi_C = 5/24, \text{and } \pi_D = 1/5$, (iv) $\begin{array}{c} Addda \\ B \\ C \end{array}\begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.6 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{pmatrix}$]

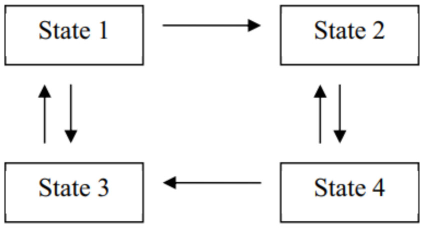
16. CT4 September 2016 Q2

The diagrams below show three Markov chains, where arrows indicate a non-zero transition probability:

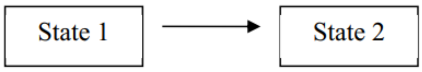
A Markov Chain 1



B Markov Chain 2



C Markov Chain 3



State whether each of the chains is:

(a) irreducible.
(b) periodic, giving the period.

Answer:

[A – Irreducible, Aperiodic, B – Irreducible, Periodic with period 2, C – Reducible, Aperiodic]

17. CT4 September 2016 Q11

An individual's marginal tax rate depends upon his or her total income during a calendar year and maybe 0% (that is, he or she is a non-taxpayer), 20% or 40%.

The movement in the individual's marginal tax rate from year to year is believed to follow a Markov Chain with a transition matrix as follows:

$$\begin{array}{r} 0\% \\ 20\% \\ 40\% \end{array}\begin{pmatrix} 1-\beta-\beta^2 & \beta & \beta^2 \\ \beta & 1-3\beta & 2\beta \\ \beta^2 & \beta & 1-\beta-\beta^2 \end{pmatrix}$$

(i) Draw the transition diagram of the process, including the transition rates.

(ii) Determine the range of values of β for which this is a valid transition matrix.

(iii) Explain whether the chain is:
(a) irreducible.
(b) periodic.
including whether this depends on the value of β.

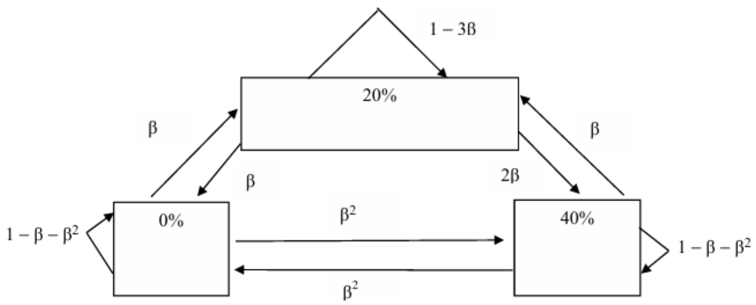
The value of β has been estimated as 0.1.

(iv) Calculate the long-term proportion of taxpayers at each marginal rate.

Lucy pays tax at a marginal rate of 20% in 2011.

(v) Calculate the probability that Lucy's marginal tax rate in 2013 is:
(a) 0%.
(b) 20%.
(c) 40%.

Answer:

[(i) 

, (ii) $0 \le \beta \le \frac{1}{3}$, (iii) irreducible and aperiodic, (iv) π_0=0.271, π_{20}=0.25, π_{40}=0.479, (v) a- 0.161, b- 0.52, c- 0.319]

18. CT4 April 2017 Q5

A city operates a bicycle rental scheme. Bicycles are stored in racks at locations around the city and may be rented for a fee and ridden from one location and deposited at another, provided there is space in the rack. The rack outside the actuarial profession's headquarters in that city has space for four bicycles.

The profession would like the city to increase the size of the rack. The city has said it will do so if the profession can demonstrate that, in the long run, the rack is full or empty for more than 35 per cent of the working day. The profession commissions a study to monitor the rack every 10 minutes during the working day.

The study shows that, on average:

- there is a probability of 0.3 that the number, m, of bicycles in the rack will increase by 1 over a 10-minute interval (where $0 \le m < 4$).
- there is a probability of 0.2 that the number of bicycles in the rack will decrease by 1 over a 10-minute interval (where $0 < m \le 4$).
- the probability of more than one increase or decrease per 10-minute interval can be regarded as 0.

(i) Give the transition matrix for the number of bicycles in the rack.

(ii) Determine whether the city will increase the size of the rack.

(iii) Comment on whether an increase in the size of the rack will reduce the proportion of time for which the rack is empty or full.

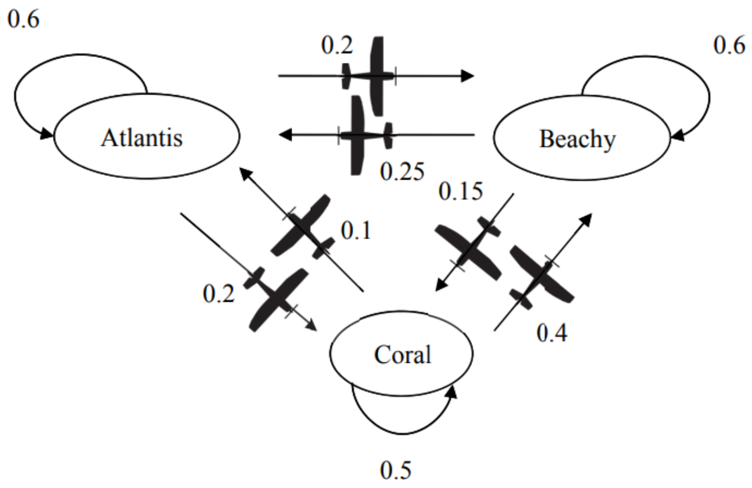
Answer:

[(i) $\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{pmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0.3 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0.3 & 0 \\ 0 & 0 & 0.2 & 0.5 & 0.3 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}$, (ii) Yes]

19. CT4 April 2017 Q9

A journalist has just been appointed as a foreign correspondent covering news stories in three island countries Atlantis, Beachy and Coral. He will spend each day in the country likely to have the most exciting news, taking the flights available between each country which go once per day at the end of the day.

The previous foreign correspondent tells him "If you want to know how many flights you are likely to take, I estimate my movements have been like this" and she drew this diagram showing the transition probabilities:



(i) Give the transition matrix for this process.

On his first day in the job the new foreign correspondent will be in Atlantis.

(ii) Calculate the probability that the foreign correspondent will be in each country in his third day in the job.

The previous correspondent also reported that Beachy must be the most interesting of the islands in terms of news because she spent 41.9% of her time there compared with 32.6% on Atlantis and 25.6% on Coral.

(iii) Sketch a graph showing the probability that the journalist is in each country over time, using the information above.

(iv) Calculate the proportion of days on which the foreign correspondent will take a flight.

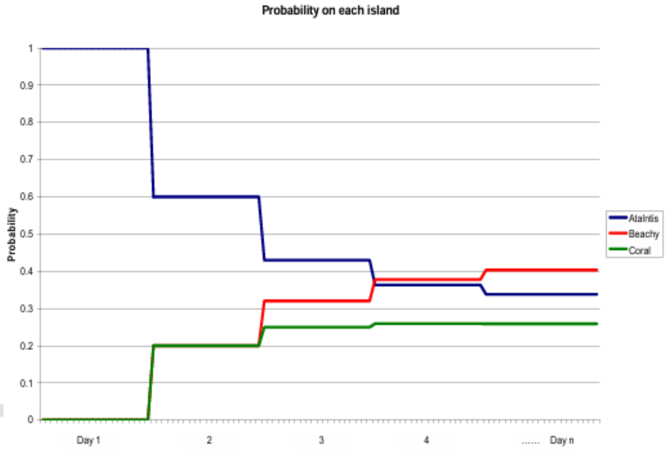
The first time the foreign correspondent visits each of the countries he takes a photograph to mark the occasion.

(v) Identify a suitable state space for modelling as a Markov chain which countries he has visited so far.

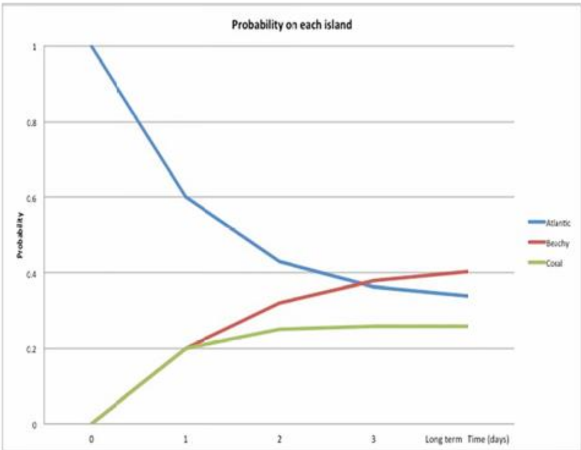
(vi) Draw a transition diagram for the possible transitions between these states.

Answer:

[(i) $\begin{array}{c} A \\ B \\ C \end{array}\begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$, (ii) 0.43-Atlantis, 0.32-Beachy, 0.25-Coral, (iii)

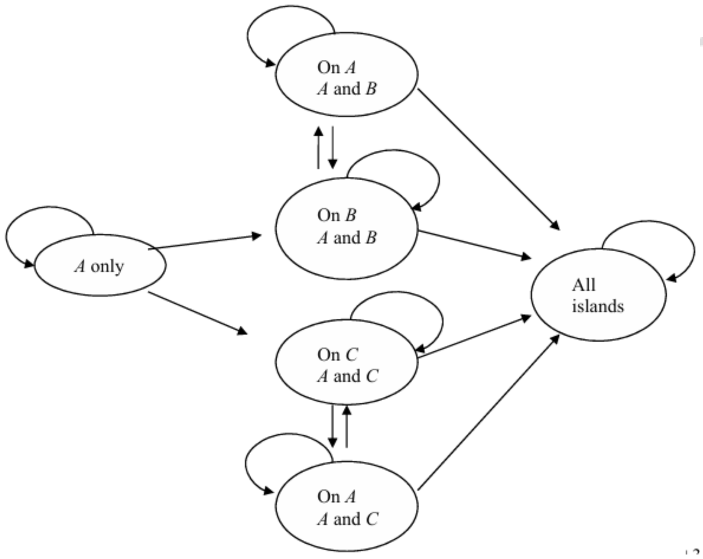


OR



,

(iv) 0 .426, (v) $\{A$ visited only, On B (A and B visited), On A (A and B visited), On C (A and C visited), On A (A and C visited), All islands visited$\}$, (vi)



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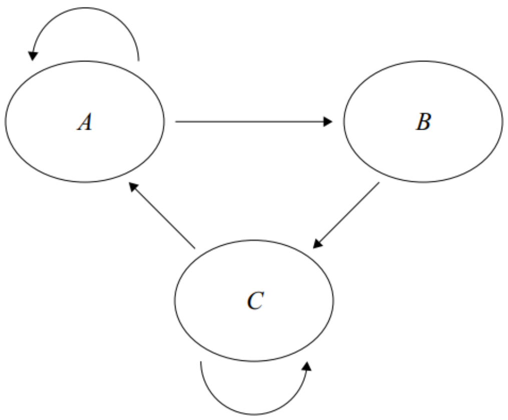
20. CT4 September 2017 Q1

A Markov Chain has the following transition graph:



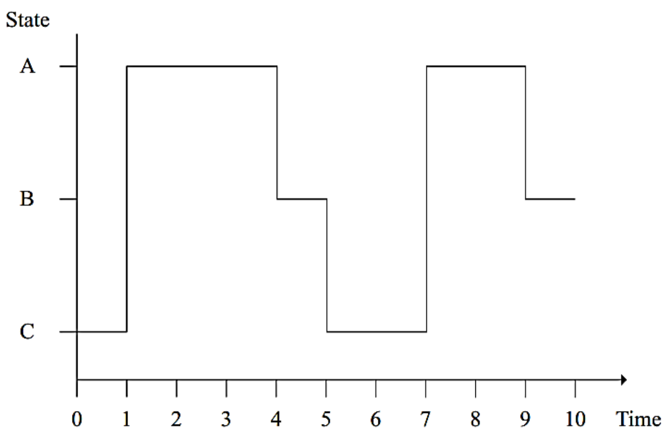
The following is a partially completed transition matrix for this Markov Chain:

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{pmatrix} 0.2 & - & - \\ - & - & 1.0 \\ - & - & 0.4 \end{pmatrix}$$

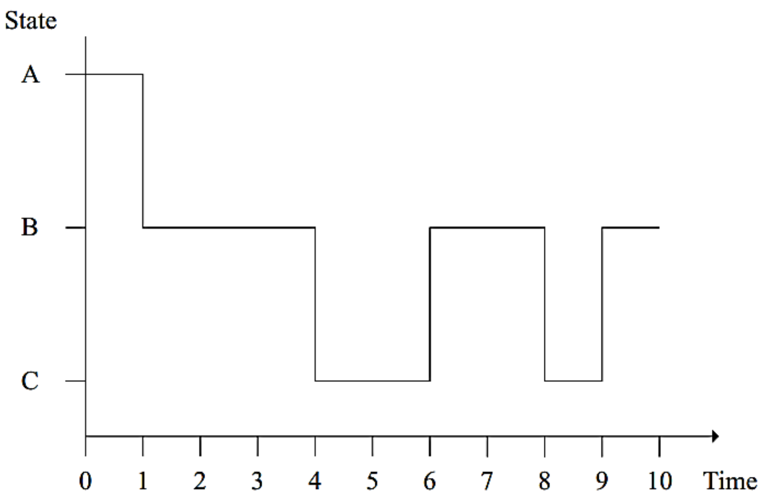
(i) Determine the remaining entries in the transition matrix.

(ii) Explain whether each of the following is a valid sample path for this process

(a) Path 1:



(b) Path 2:



Answer:

[(i) $\begin{array}{c} A \\ B \\ C \end{array}\begin{pmatrix} 0.2 & 0.8 & 0 \\ 0 & 0 & 1.0 \\ 0.6 & 0 & 0.4 \end{pmatrix}$. , (ii) a – yes, b - no]

21. CT4 September 2017 Q8

A company has for many years offered a car insurance policy with four levels of No Claims Discount (NCD): 0%, 15%, 30% and 40%. A policyholder who does not claim in a year moves up one level of discount, or remains at the highest level. A policyholder who claims one or more times in a year moves down a level of discount or remains at the lowest level. The company pays a maximum of three claims in any year on any one policy.

The company has established that:

- the arrival of claims follows a Poisson process with a rate of 0.35 per year.
- the average cost per claim is £2,500.
- the proportion of policyholders at each level of discount is as follows:

Discount level	*Proportion of policyholders*
0%	4.4%
15%	10.5%
30%	25.1%
40%	60.0%

(i) Calculate the premium paid by a policyholder at the 40% discount level ignoring expenses and profit.

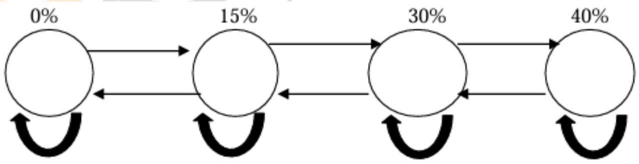
The company has decided to introduce a protected NCD feature whereby policyholders can make one claim on their policy in a year and, rather than move down a level of discount, remain at the level they are at. All other features of the policy remain the same.

(ii) Draw the transition graph for this process.

(iii) Calculate the premium paid, in the long term, by a policyholder at the 40% discount level of the policy with protected NCD, ignoring expenses and profit.

(iv) Discuss THREE issues with the policy with protected NCD which may each be either a disadvantage or an advantage to the company.

Answer:

[(i) £783.67, (ii) 

, (iii) £862.71]

22. CT4 April 2018 Q1

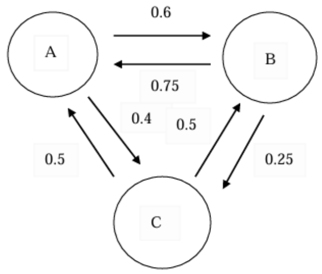
A Markov Chain has the following transition matrix:

$$\begin{array}{c} A \\ B \\ C \end{array} \begin{pmatrix} 0 & 0.6 & 0.4 \\ 0.75 & 0 & 0.25 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

(i) Draw a transition graph for this Markov Chain, including the transition rates.

(ii) Explain whether the Markov Chain is:
(a) irreducible
(b) periodic

Answer:

[(i) 

, (ii) irreducible & aperiodic]

23. CT4 April 2018 Q7

(i) Define a Markov Chain.

The manager of a sales team keeps records of how much each of the three sales staff (Andy, Brenda and Carol) sells each week. The data suggests that the sales staff member who makes the most sales each week can be modelled using a Markov Chain with the following transition matrix:

$$\begin{array}{c} \textit{Andy} \\ \textit{Brenda} \\ \textit{Carol} \end{array} \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.5 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

Brenda made the most sales in the first week in April.

(ii) Calculate the probability that each member of the sales staff makes the most sales in the third week of April.

(iii) Calculate the long-term proportion of weeks in which each member of the sales staff makes the most sales.

The manager is keen to encourage competition in the team, so he introduces an "Employee of the Week" incentive. He awards "Employee of the Week" to the member of the sales staff who makes the most sales unless this is the same employee who was awarded "Employee of the Week" last week. If last week's "Employee of the Week" makes the most sales the manager will decide which of the other two staff should be "Employee of the Week" and is equally likely to choose either.

(iv) Justify why whoever is awarded "Employee of the Week" can NOT be modelled as a Markov Chain with state space {Andy, Brenda, Carol}.

(v) Identify a state space with the minimum number of states required to model the sequence of "Employees of the Week" as a Markov Chain.

Answer:

[(ii) Andy - 0.31, Brenda - 0.40, Carol - 0.29, (iii) Andy - 0.2969, Brenda - 0.375, Carol - 0.3281]

24. CT4 September 2018 Q1

Explain why each of the following matrices is, or is not, a valid transition matrix for a Markov chain.

(a) $\begin{pmatrix} 0 & 0 \\ 0.5 & 0.5 \end{pmatrix}$

(b) $\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.1 & 0.6 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0.8 \\ 0.4 & 0.6 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & -0.1 & 0.7 \end{pmatrix}$

Answer:

[a – Not valid, b - Not valid, c – valid, d - Not valid]

25. CT4 September 2018 Q12

A small town is served by a single funeral director. The funeral director collects corpses immediately following death and stores them in a refrigerator pending embalming. The number of deaths per day in this town has the following probability distribution:

Number of deaths per day	*Probability*
0	0.497
1	0.348
2	0.122
3	0.028
4	0.005

The embalmer can embalm exactly one corpse per day. He works on a corpse from the refrigerator if there is one, but if the refrigerator is empty he works on the first corpse to arrive that day. Corpses are removed from the refrigerator immediately before being embalmed and are not returned there after embalming.

The refrigerator has room for four corpses. If more space is needed, the funeral director has to ask the local hospital if there is spare capacity in the hospital's refrigerator.

(i) Determine the transition matrix for the number of corpses in the funeral director's refrigerator.

(ii) Calculate the long-run probability of there being 0, 1, 2, 3 and 4 corpses in the refrigerator.

(iii) Calculate the probability that the funeral director has to contact the hospital on any given day.

The embalmer has not had a day off for years. The funeral director says that from now on the embalmer must not work on Christmas Day.

(iv) Calculate the probability that the funeral director will need to contact the hospital on Christmas Day when the embalmer is not working.

Answer:

[(i) $$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 0.845 & 0.122 & 0.028 & 0.005 & 0 \\ 0.497 & 0.348 & 0.122 & 0.028 & 0.005 \\ 0 & 0.497 & 0.348 & 0.122 & 0.033 \\ 0 & 0 & 0.497 & 0.348 & 0.155 \\ 0 & 0 & 0 & 0.497 & 0.503 \end{bmatrix}$$, (ii) x0 = 0.626, x1 = 0.195, x2 = 0.103, x3 = 0.051, x4 = 0.025, (iii) 0.006, (iv) 0.025]

26. CS2A September 2019 Q4

(i) State the Markov property.

The three transition matrices below describe three Markov Chains.

I

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

II

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.75 & 0.25 \end{bmatrix}$$

III

$$\begin{bmatrix} 0.25 & 0.75 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.75 & 0.25 \end{bmatrix}$$

(ii) State whether these Markov Chains are:
(a) irreducible
(b) periodic, stating the period where appropriate.

(iii) Calculate the stationary probability distribution (if it exists) for each of the three Markov Chains described above.

Answer:

[(ii)a - I irreducible, II reducible, III irreducible, b - I periodic, II aperiodic, III aperiodic, (iii) I {0.5, 0.5}, II {1, 0, 0, 0}, III {0.2, 0.3, 0.3, 0.2}]

27. CS2A September 2020 Q7

The transition matrix, P, describes a Markov chain for the state space S = {1, 2, 3, 4}:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Explain whether this Markov chain is irreducible.

(ii) Determine all of the stationary probability distributions, π, for the Markov chain defined above, if any such probability distributions exist.

The hitting probability, hij, is defined as the probability of ever reaching state j, starting from the initial state i.

(iii) Determine the hitting probabilities for state 4 (hi4) for all states i in S.

Answer:

[(i) not irreducible, (ii) pi = {x, 0, 0, 1 – x} where 0 ≤ x ≤ 1, (iii) h_i4 = {0, 1/3, 2/3, 1}]

28. CS2A September 2020 Q9

An insurance company offers annual home insurance policies in partnership with a bank. The distribution deal involves taking part in the bank's loyalty scheme called '1234'. Under '1234', a customer gets a discount when buying or renewing the policy according to how many bank accounts they hold, as follows:

Number of bank accounts	*Discount*
One	5%
Two	10%
Three	15%
Four or more	25%

An analysis of the data suggests that the transition matrix for the number of bank accounts held at annual intervals is as follows:

$$\begin{array}{l} \text{One} \\ \text{Two} \\ \text{Three} \\ \text{Four+} \end{array} \begin{pmatrix} 0.5 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0.4 & 0.2 \\ 0 & 0.2 & 0.2 & 0.6 \end{pmatrix}$$

A customer takes out a policy in January 2017 at the 10% discount level.

(i) Calculate the probability that the customer remains at the 10% discount level for all of their renewals up to and including 2020.

(ii) Calculate the probability that the customer receives a discount of at least 15% in 2019.

(iii) Which one of the following options represents the correct stationary distribution, π, of the transition matrix above?

A	$\pi_1 = \frac{34}{45}\pi_3,$	$\pi_2 = \frac{8}{9}\pi_3,$	$\pi_3,$	$\pi_4 = \frac{41}{45}\pi_3$
B	$\pi_1 = \frac{34}{45}\pi_3,$	$\pi_2 = \frac{8}{9}\pi_3,$	$\pi_3,$	$\pi_4 = \pi_3$
C	$\pi_1 = \pi_3,$	$\pi_2 = \frac{8}{9}\pi_3,$	$\pi_3,$	$\pi_4 = \frac{41}{45}\pi_3$
D	$\pi_1 = \pi_3,$	$\pi_2 = \pi_3,$	$\pi_3,$	$\pi_4 = \pi_3$

(iv) Which one of the following options represents the correct average long-term level of discount?

A 13.60%
B 13.75%
C 14.19%
D 14.45%

(v) Comment on the commercial implications for the insurer of the bank's loyalty scheme.

Answer:
[(i) 0.064, (ii) 0.48, (iii) A, (iv) C]

29. CS2A September 2021 Q9

In a small country, there are only four authorised car insurance companies A, B, C and D. All car owners take out car insurance from an authorised insurance company. All policies provide cover for a period of 1 calendar year.

The probabilities of car owners transferring between the four companies at the end of each year are believed to follow a Markov chain with the following transition matrix:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{pmatrix} 1-2\alpha-\alpha^2 & \alpha & \alpha & \alpha^2 \\ \alpha & \frac{1}{2}-\alpha & \frac{1}{2}-\alpha & \alpha \\ \alpha & \frac{1}{2}-\alpha & \frac{1}{2}-\alpha & \alpha \\ 0 & \alpha & \alpha & 1-2\alpha \end{pmatrix}$$

for some parameter α.

(i) Determine the range of values of α for which this is a valid transition matrix.

(ii) Explain whether the Markov chain is irreducible, including whether this depends on the value of α.

The value of α has been estimated as 0.2.

Mary has just bought her policy from Company D for the first time.

(iii) Determine the probability that Mary will be covered by Company D for at least 4 years before she transfers to another insurance company.

James took out a policy with Company A in January 2018. Sadly, James' car was stolen on 23 December 2020.

(iv) Determine the probability that a different company, other than Company A, covers James' car at the time it was stolen.

Company A makes an offer to buy Company D. It bases its purchase price on the assumption that car owners who would previously have purchased policies from Company A or Company D would now buy from the combined company, to be called ADDA.

(v) Write down the transition matrix that will apply after the takeover if Company A's assumption about car owners' behaviour is correct.

(vi) Comment on the appropriateness of Company A's assumption.

Answer:

[(i) 0 <= alpha <= sqrt(2) – 1, (iii) 0.216, (iv) 0.6064, (v)

ADDA	0.6	0.2	0.2
B	0.4	0.3	0.3
C	0.4	0.3	0.3

]

30. CS2A April 2022 Q8

The number of customers, N_t, in a queue at each integer time t is modelled using a Markov chain model.

At the start of each time interval, a number of customers following a Poisson distribution with parameter p join the queue, subject to the constraint that the number of customers in the queue can be no more than N_{max}.

At the end of each time interval, a number of customers following a Poisson distribution with parameter q, where q > p, are served and leave the queue, subject to the constraint that the number of customers in the queue cannot be negative.

(i) Comment on the limitations of this model.

An analyst sets the model's parameters to N_{max} = 2, p = 0.5 and q = 1.

(ii) Verify, by separately considering the two transition matrices of customers joining the queue and customers leaving the queue, respectively, that the transition matrix of N_t is:

$$\begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{pmatrix} 0.82207 & 0.14475 & 0.03318 \\ 0.48737 & 0.36788 & 0.14475 \\ 0.26424 & 0.36788 & 0.36788 \end{pmatrix}$$

(iii) Determine the stationary distribution of N_t.

Answer:

[(iii) Stationary distribution is (0.7040, 0.2108, 0.0852)]