INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical and risk Modelling 3

Chapter: Unit 3 & 4

Category: Practice Questions

1. CT4 April 2015 Q10

In a computer game a player starts with three lives. Events in the game which cause the player to lose a life occur with a probability $\mu dt + o(dt)$ in a small-time interval dt. However the player can also find extra lives. The probability of finding an extra life in a small-time interval dt is $\lambda dt + o(dt)$. The game ends when a player runs out of lives.

(i) Outline the state space for the process which describes the number of lives a player has.

(ii) Draw a transition graph for the process, including the relevant transition rates.

(iii) Determine the generator matrix for the process.

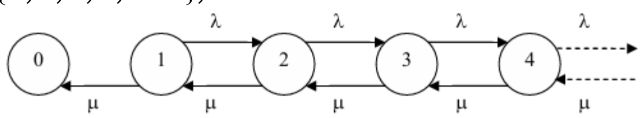
(iv) Explain what is meant by a Markov jump chain.

(v) Determine the transition matrix for the jump chain associated with the process.

(vi) Determine the probability that a game ends without the player finding an extra life.

Answer:

[(i) {0,1,2,3,4....},

(ii) 

,

(iii)
$$\begin{array}{c} \text{Lives} \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & . \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) & \\ . & & & & & . \end{pmatrix} \end{array},$$

(v)
$$\begin{array}{c} \text{Lives} \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \text{etc.} \\ \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 & \\ 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & \\ 0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & \\ 0 & 0 & 0 & \mu/(\mu+\lambda) & 0 & \\ \text{etc.} & & & & & \end{pmatrix} \end{array},$$

(vi) $\left(\dfrac{\mu}{\mu+\lambda}\right)^3$]

2. CT4 April 2015 Q11

A new disease has been discovered which is transmitted by an airborne virus. Anyone who contracts the disease suffers a high fever and then in 60% of cases dies within an hour and in 40% of cases recovers. Having suffered from the disease once, a person builds up antibodies to the disease and thereafter is immune.

(i) Draw a multiple state diagram illustrating the process, labelling the states and possible transitions between states.

(ii) Express the likelihood of the process in terms of the transition intensities and other observable quantities, defining all the terms you use

(iii) Derive the maximum likelihood estimator of the rate of first time sickness.

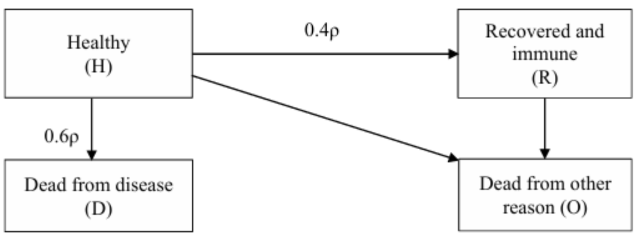
Three years ago medical students visited the island where the disease was first discovered and found that of the population of 2,500 people, 860 had suffered from the disease but recovered. They asked the leaders of the island to keep records of the occurrence and the outcome of each incidence of the disease. The students intended to return exactly three years later to collect the information.

(iv) Derive an expression (in terms of the transition intensities) for the probability that an islander who has never suffered from the disease will still be alive in three years' time.

(v) Set out the information which the students would need when they returned three years later in order to calculate the rate of sickness from the disease.

Answer:

[(i)



,

(ii) $$L \propto \exp\left\{\left(-0.6\rho - 0.4\rho - \mu^{HO}\right)v^{H}\right\}\exp\left\{\left(-\mu^{RO}\right)v^{R}\right\}\left(0.6\rho\right)^{d^{HD}}\left(0.4\rho\right)^{d^{HR}}\left(\mu^{HO}\right)^{d^{HO}}\left(\mu^{RO}\right)^{d^{RO}}$$

, (iii) $$\hat{\rho} = \frac{d^{HD} + d^{HR}}{v^{H}}$$,

(iv) $\int_0^3 \exp\left[-\left\{\left(\rho+\mu^{HO}\right)u\right\}\right]0.4\rho\exp\left[-\left\{\left(\mu^{RO}\right)(3-u)\right\}\right]du$.
]

3. CT4 September 2015 Q8

(i) Define a Markov Jump Process.

A company provides phones on contracts under which it is responsible for repairing or replacing any phones which break down.

When a customer reports a fault with a phone, it is immediately taken to the company's repair shop and it is assessed whether it can be fixed (meaning fixable at reasonable cost). Based on previous experience, it is estimated that the probability of a phone being fixable is 0.75. If a phone is not fixable it is discarded and the customer is provided with a new phone.

If a repaired phone breaks again the company, in line with its customer charter, will not attempt to repair it again, and so discards the phone and replaces it with a new one.

The status of a phone is to be modelled as a Markov Jump Process with state space { Never Broken (NB), Repaired (R), Discarded (D) }.

The company considers the rate at which phones break down to vary according to whether a phone has previously been repaired as follows:

Status	Probability of break down in small interval of time, dt
Never Broken	$0.1dt + o(dt)$
Repaired	$0.2dt + o(dt)$

(ii) Draw a transition diagram for the possible transitions between the states, including the associated transition rates.

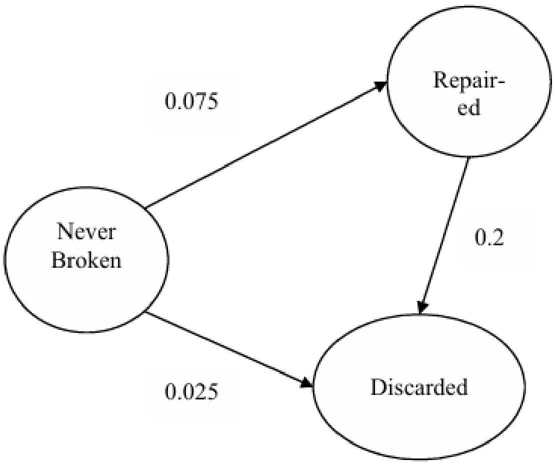
Let $P_{NB}(t), P_R(t)$ and $P_D(t)$ be the probabilities that a phone is in each state after time t since it was provided as a new phone.

(iii) Determine Kolmogorov's forward equations in component form for $P_{NB}(t)$, $P_R(t)$ and $P_D(t)$.

(iv) Solve the equations in part (iii) to obtain $P_{NB}(t)$ and $P_R(t)$.

(v) Calculate the probability that a phone has not been discarded by time t.

Answer:



[(ii) ,

$$\frac{d}{dt}P_{NB}(t) = -0.1P_{NB}(t)$$

$$\frac{d}{dt}P_R(t) = 0.075P_{NB}(t) - 0.2P_R(t)$$

$$\frac{d}{dt}P_D(t) = 0.2P_R(t) + 0.025P_{NB}(t)$$

(iii) ,

(iv) $P_{NB}(t) = \exp(-0.1t)$, $P_R(t) = 0.750\ (\exp(-0.1t) - \exp(-0.2t))$, (v) $1.75\exp(-0.1t) - 0.75\exp(-0.2t)$]

4. CT4 April 2016 Q7

(i) State the condition needed for a Markov Jump Process to be time inhomogeneous.

(ii) Describe the principal difficulties in modelling using a Markov Jump Process with time inhomogeneous rates.

A multi-tasking worker at a children's nursery observes whether children are being "Good" or "Naughty" at all times. Her observations suggest that the probability of a child moving between the two states varies with the time, t, since the child arrived at the nursery in the morning. She estimates that the transition rates are:

From Good to Naughty: $0.2 + 0.04t$
From Naughty to Good: $0.4 - 0.04t$

where t is measured in hours from the time the child arrived in the morning, $0 \le t \le 8$.

A child is in the "Good" state when he arrives at the nursery at 9 a.m.

(iii) Calculate the probability that the child is Good for all the time up until time t.

(iv) Calculate the time by which there is at least a 50% chance of the child having been Naughty at some point.

Let P_G(t) be the probability that the child is Good at time t.
(v) Derive a differential equation just involving PG(t) which could be used to determine the probability that the child is Good on leaving the nursery at 5 p.m.

Answer:

[(iii) $P_{\overline{GG}}(t) = \exp(-0.2t - 0.02t^2)$., (iv) 2.724., (v) $\frac{d}{dt}P_G(t) = 0.4 - 0.04t - 0.6P_G(t)$.]

5. CT4 April 2016 Q9

Orange trees are susceptible to the disease Citrus Greening. There is no known cure for this disease and, although trees often survive for some time with the disease, it can ultimately be fatal.

A researcher decides to model the progression of the disease using a time-homogeneous continuous-time Markov model with the following state space:

{Healthy (i.e. not infected with Citrus Greening);
Infected with Citrus Greening;
Dead (caused by Citrus Greening);
Dead (other causes)}.

The researcher chooses to label the transition rate parameters as follows:

- a mortality rate from the Healthy state, μ
- a rate of infection with Citrus Greening, σ
- a total mortality rate from the Infected state, ρ
- a mortality rate caused by Citrus Greening, τ

(i) Draw a transition diagram for the chosen model, including the transition rates.

(ii) Determine Kolmogorov's forward equations governing the transitions, specifying the generator matrix.

Infected trees display clear symptoms of the disease. This has enabled the researcher to record the following data on trees in the area of his study:

Tree-months in Healthy State	1,200
Tree-months in Infected State	600
Total number of deaths of trees	40
Number of deaths of Healthy trees	10
Number of deaths from Citrus Greening	30

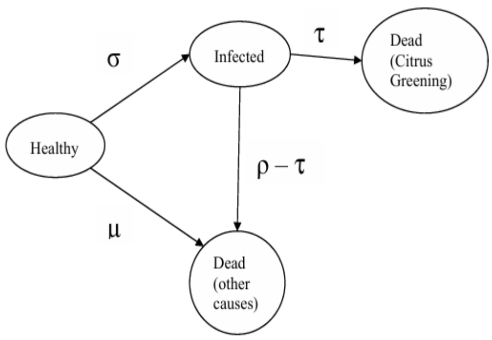
(iii) Give the likelihood of these data.

(iv) Derive the maximum likelihood estimator of the mortality rate caused by Citrus Greening, τ.

(v) Estimate τ.

Answer:

[(i)



, (ii)

$$\frac{d}{dt}P(t) = P(t)A,$$

where A is the generator matrix:

$$A = \begin{pmatrix} -\sigma-\mu & \sigma & 0 & \mu \\ 0 & -\rho & \tau & \rho-\tau \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with the order of states being {Healthy, Infected, Dead (caused by Citrus Greening), Dead (other causes)}.

, (iii)

$$L \propto \exp\left[(-\mu-\sigma)v^H\right]\exp\left[(-\xi-\tau)v^I\right](\sigma)^{d^{HI}}(\mu)^{d^{HDOC}}(\tau)^{d^{IDCG}}(\xi)^{d^{IDOC}},$$

(iv) $\hat{\tau} = \dfrac{d^{IDCG}}{v^I}$. OR $\hat{\tau} = \dfrac{\rho d^{IDCG}}{d^{IDOC} + d^{IDCG}}$. ,

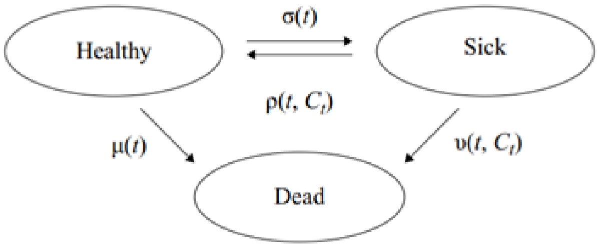
(v) 0.05]

6. CT4 September 2017 Q7

The following diagram shows the transitions under a Healthy-Sick-Dead multiple state model under which:

• transition rates are dependent on time, t.

• transitions out of the Sick state are dependent on the duration, Ct, a person has been in the Sick state as well as on time



(i) Show from first principles, that if $p_{ij}(x,t)$ is the probability of being in state j at time t conditional on being in state i at time x. that

$$\frac{\partial}{\partial t} p_{HH}(x,t) = p_{HH}(x,t)(-\sigma(t)-\mu(t)) + p_{HS}(x,t)\rho(t,C_t)$$

(ii) Determine the probability that a life is in the Healthy state throughout the period 0 to t if the life is in the Healthy state at time 0.

(iii) Describe how integrated Kolmogorov equations can be constructed by conditioning on the first or the last jump, illustrating your answer with a diagram.

(iv) Explain the difference in approach between deriving forward and backward integrated Kolmogorov equations.

An actuarial student suggests the following integrated Kolmogorov equation for this model:

$$\Pr[X_t = H \mid X_s = S, C_s = w] = \int_s^t e^{-\int_s^y (\rho(u,w-s+u)+\upsilon(u,w-s+u))du} \rho(y, w-s+y) P_{HH}(y,t)dy$$

(v) Identify TWO errors in this equation

UNIT 3 & 4

PRACTICE QUESTIONS

Answer:

[(ii) $$p_{\overline{HH}}(0,t) = \exp -\left(\int_{s=0}^{t} (\sigma(s) + \mu(s))\, ds\right)$$]

7. CT4 April 2018 Q5

The calculation of the daily unit price for a fund investing in commercial properties can be done on one of two bases:

• Bid price – reflecting the price at which the properties could be sold, allowing for the transaction costs for selling properties.

• Offer price – reflecting the price at which properties could be purchased, again allowing for the transaction costs which would apply.

Whether the fund is priced on the Bid or Offer basis depends on whether there is net investment into or redemption from the fund.

Movements between the states Bid (B) and Offer (O) pricing basis are to be modelled using a Markov Jump Process with constant transition rates from Bid to Offer of l and Offer to Bid of μ.

(i) Give the generator matrix of the Markov Jump Process.

(ii) State the distribution of holding times in each state.

Let ${}_tP_s^{ij}$ be the probability that the process is in state j at time $s+t$ given that it was in state i at time $s(i,j = B,O)$.

(iii) Write down Kolmogorov's forward equations for $\frac{d}{dt}\,{}_tP_s^{BB}$ and $\frac{d}{dt}\,{}_tP_s^{BO}$.

The fund was being priced on a Bid price basis at time s.

(iv) Solve the Kolmogorov equations to obtain an expression for ${}_tP_s^{BB}$.

Answer:

[(i) $\begin{array}{c} B \\ O \end{array}\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$,

(ii) exponentially distributed with parameter λ in state B, and μ in state O,

(iii) $\frac{\partial}{\partial t}\,{}_tP_s^{BB} = -\lambda\,{}_tP_s^{BB} + \mu\,{}_tP_s^{BO}$

$\frac{\partial}{\partial t}\,{}_tP_s^{BO} = \lambda\,{}_tP_s^{BB} - \mu\,{}_tP_s^{BO}$, (iv) ${}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}.\exp(-(\lambda+\mu)t)$]

8. CT4 September 2018 Q6

A Markov jump process has the following generator matrix:

$$\begin{array}{c} A \\ B \\ C \end{array}\begin{pmatrix} -0.3 & 0.2 & 0.1 \\ 0.1 & -0.5 & 0.4 \\ 0.3 & 0.1 & -0.4 \end{pmatrix}$$

(i) Draw a transition graph for this process.

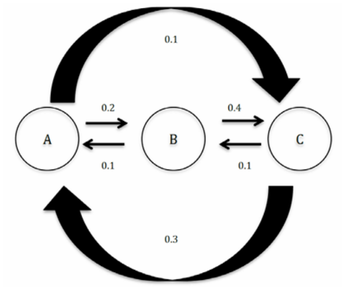
The process is in state A at time zero.

(ii) Give Kolmogorov's forward equations for $\frac{d}{dt}P_{AA}(t), \frac{d}{dt}P_{AB}(t)$ and $\frac{d}{dt}P_{AC}(t)$.

(iii) Calculate the probability that the process remains in state A throughout the period $t = 0$ to $t = 2$.

(iv) Determine the probability that the third jump of the process is into state C.

Answer:

[(i) 

, (ii)

$$\frac{d}{dt}P_{AA}(t) = -0.3P_{AA}(t) + 0.1P_{AB}(t) + 0.3P_{AC}(t)$$

$$\frac{d}{dt}P_{AB}(t) = 0.2P_{AA}(t) - 0.5P_{AB}(t) + 0.1P_{AC}(t)$$

$$\frac{d}{dt}P_{AC}(t) = 0.1P_{AA}(t) + 0.4P_{AB}(t) - 0.4P_{AC}(t)$$

,

(iii) 0.5488, (iv) 0.194]

9. CT4 September 2018 Q8

A drug named Nimble is often prescribed to the elderly for periods greater than a year to reduce the pain of arthritis. It has been proven that taking Nimble increases the chance of death by heart disease. One local health authority is looking into whether the increased risk continues once a patient stops taking the drug. It proposes to use a model with five states: (1) Never taken Nimble, (2) Taking Nimble, (3) No longer taking Nimble, (4) Dead through heart disease, (5) Dead through other causes.

(i) Draw a diagram showing the possible transitions between the five states.

Let the transition intensity between state i and state j at time $x+t$ be $\mu^{ij}{}_{x+t}$. Let the probability that a person in state i at time x will be in state j at time $x+t$ be ${}_tp_x{}^{ij}$.

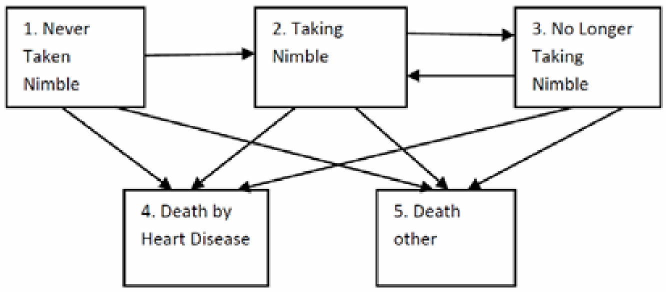
(ii) Show from first principles that

$$\frac{d}{dt}\left({}_tp_x^{34}\right) = {}_tp_x^{32}\mu_{x+t}^{24} + {}_tp_x^{33}\mu_{x+t}^{34}$$

The health authority wishes to calculate death rates aged 50 last birthday from each cause for individuals in states 1, 2 and 3.

(iii) State what data would need to be extracted from the health authority's medical records in order to do this.

Answer:

[(i) 

]

10. CS2A April 2021 Q4

A duration-dependent Markov jump process model is constructed to model the recovery and mortality rates of individuals who are sick with a particular medical condition. The model includes the following three states:

- sick
- recovered
- dead.

Under this model, an individual who has been sick for duration t experiences:

- a transition rate to the recovered state of a.exp(–bt)
- a transition rate to the dead state of a(1– exp(–bt)) + c

where a, b and c are parameters > 0 that depend on the characteristics of the individual.

(i) Comment on the reasonableness of this model.

(ii) Demonstrate that the probability that an individual who has just become sick eventually recovers is:

$$\frac{a}{(a+b+c)}.$$

11. CS2A April 2022 Q3

During a football match, the referee can caution players if they commit an offence by showing them a yellow card. If a player commits a second offence that the referee deems worthy of a caution, they are shown a red card, and are sent off the pitch and take no further part in the match. If the referee considers a particularly serious offence to have been committed, a red card will be shown to a player who has not previously been cautioned, and the player will be sent off immediately.

The football team manager can also decide to substitute one player for another at any point in the match so that the substituted player takes no further part in the match. Due to the risk of a player being sent off, the manager is more likely to substitute a player who has been shown a yellow card. Experience shows that players who have been shown a yellow card play more carefully to try to avoid a second offence.

- The rate at which uncautioned players are shown a yellow card is 1/8 per hour.
- The rate at which players who have already been shown a yellow card are shown a red card is 1/12 per hour.
- The rate at which uncautioned players are shown a red card is 1/20 per hour.
- The rate at which players are substituted is 1/10 per hour if they have not been shown a yellow card, and 1/5 if they have been shown a yellow card.

An actuary decides to model a player's state during a match using a Markov jump process and the following four states:

- State U: uncautioned
- State Y: yellow card shown
- State R: red card shown
- State S: substituted.

(i) Write down the generator matrix of the Markov jump process that is used in the compact form of Kolmogorov's forward equations.

A football match lasts 1.5 hours.

(ii) Solve Kolmogorov's forward equation for the probability that a player who starts the match remains in the game for the whole match without being substituted and without being shown a yellow card or a red card.

(iii) Determine the probability that a player who starts the match is sent off during the match without having previously been cautioned.

Answer:

[(i) Generator matrix A of Markov jump process =

	U	Y	R	S
U	– 11 / 40	1 / 8	1 / 20	1 / 10
Y	0	– 17 / 60	1 / 12	1 / 5
R	0	0	0	0
S	0	0	0	0

,
(ii) 66.20%, (iii) 6.15%]

12. CS2A April 2023 Q2

Country A has recently gone through an economic crisis. As the country makes an attempt to recover, the Department of Finance is trying to estimate the rate of recovery in employment. The department has decided to use a two-state continuous-time Markov model to estimate the rate of return to employment. It has also decided to use data from one of the previous economic recoveries for the purpose. The two states are:

$$\boxed{\text{Unemployed}} \longrightarrow \boxed{\text{Employed}}$$

The employment data from a previous economic recovery was as follows:

- Waiting time to gain employment in the first year (in person-years): 30,000
- Waiting time to gain employment in the second year (in person-years): 22,000
- Number of people gaining employment in the first year: 5,000
- Number of people gaining employment in the second year: 7,000

It may be assumed that force of gaining employment in any 1 year is constant.

(i) State the likelihood function of the maximum likelihood estimator of the transition rate defining all the terms you use.

(ii) Calculate the maximum likelihood estimate of the transition rate from the state of being unemployed to gaining employment for each of the first 2 years.

(iii) Estimate, by stating the expression, the variance of the second year maximum likelihood estimator.

(iv) Calculate the probability of not gaining any employment in the next 2 years.

Answer:

[(i) L (*mu*_i;di,vi) = exp(-*mu*_i *vi)* *mu*_i ^di , (ii) 0.16667, 0.31818, (iii) 0.000014, (iv) 0.61579]

13. CS2A April 2023 Q4

A Markov jump process model is used to describe the recovery of people bitten by a certain type of poisonous snake. There are three states:

- Sick and receiving medical care following the snake bite
- Fully recovered
- Recovered but with long-term health effects from the bite.

(i) Explain why a time in-homogeneous Markov jump process model is more suitable than a simpler time homogeneous multi-state model in this scenario.

The transition rates from the sick state in this model t days after being bitten by the snake are:

$e^{-2.5t}$	for the transition to fully recovered and
$0.05 - e^{-2.5t}$	for the transition to recovered but with long-term health effects.

(ii) Comment on the key features of this model including the transition rates.

(iii) Determine the probability that a person just bitten by a snake will eventually make a full recovery without any long-term health effects.

Answer:

[(iii) = 0.392]

14. CS2A September 2024 Q4

An insurance company has a portfolio of 350 policies and assumes that the policies exist in three possible states of claim costs: low, moderate and high. For each of these states the overall claim sizes are assumed to be £200, £300 and £500, respectively.

The policies change states according to a Markov jump process with the generator matrix:

$$A = \begin{pmatrix} * & 0.1 & 0.5 \\ 0.3 & * & 0.4 \\ 0.1 & * & -0.2 \end{pmatrix}$$

(i) State the missing entries A11, A22 and A32 marked * in the generator matrix A above.

(ii) Derive the transition matrix, P, for the corresponding Markov jump process.

(iii) Derive the stationary distribution related to the matrix P.

(iv) Calculate the expected value and the standard deviation of the total claims for this portfolio based on the distribution in part (iii).

Answer:

[(i) $A = \begin{pmatrix} -\mathbf{0.6} & 0.1 & 0.5 \\ 0.3 & -\mathbf{0.7} & 0.4 \\ 0.1 & \mathbf{0.1} & -0.2 \end{pmatrix}$, (ii) $P = \begin{pmatrix} 0 & \frac{0.1}{0.6} & \frac{0.5}{0.6} \\ \frac{0.3}{0.7} & 0 & \frac{0.4}{0.7} \\ \frac{0.1}{0.2} & \frac{0.1}{0.2} & 0 \end{pmatrix}$, (iii) $\pi = (0.3208556, 0.2620321, 0.4171123)$,
(iv) Expected cost =122967.9 & Std deviation = 45978.83]