INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical Techniques and Risk Modelling IV

Chapter: Unit 3

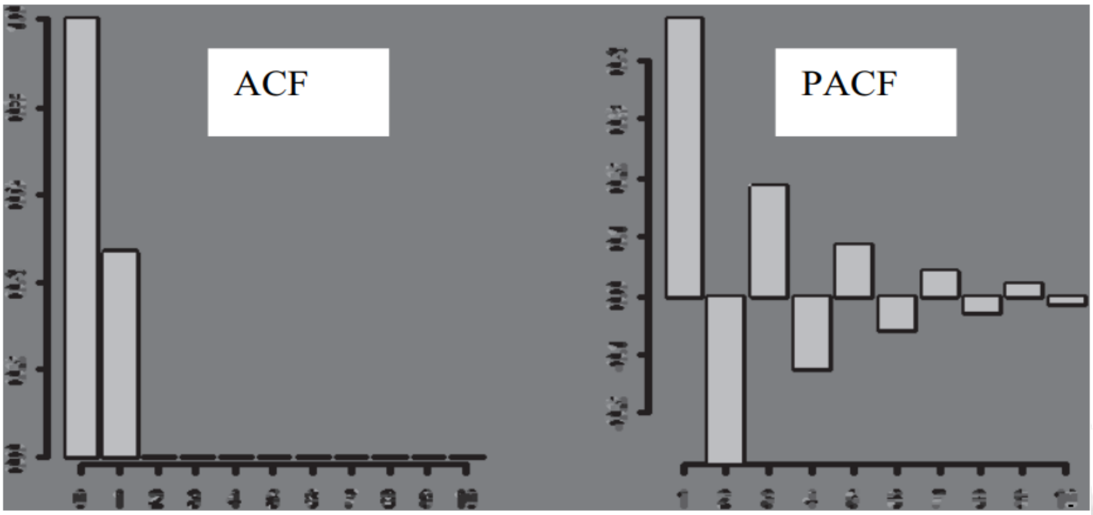
Category: Practice Questions

ALL Questions are IFOA Past Paper Questions

1. Subject CT6 September 2013

(i) State the three main stages in the Box-Jenkins approach to fitting an ARIMA time series model.

(ii) Explain, with reasons, which ARIMA time series would fit the observed data in the charts below.



Now consider the time series model given by

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \beta_1 e_{t-1} + e_t$$

where e_t is a white noise process with variance σ^2.

(iii) Derive the Yule-Walker equations for this model.

(iv) Explain whether the partial auto-correlation function for this model can ever give a zero value.

2. Subject CT6 April 2014 Question 12

A sequence of 100 observations was made from a time series and the following values of the sample auto-covariance function (SACF) were observed:

Lag SACF

1 0.68
2 0.55
3 0.30
4 0.06

The sample mean and variance of the same observations are 1.35 and 0.9 respectively.

(i) Calculate the first two values of the partial correlation function $\hat{\phi}_1$ *and* $\hat{\phi}_2$

(ii) Estimate the parameters (including σ2) of the following models which are to be fitted to the observed data and can be assumed to be stationary.

(iii)
(a) Y_t = a0 + a1 Yt−1 + et

(b) Y_t = a0 + a1 Yt−1 + a2 Yt−2 + et

In each case et is a white noise process with variance σ^2.

(iv) Explain whether the assumption of stationarity is necessary for the estimation for each of the models in part (ii).

(v) Explain whether each of the models in part (ii) satisfies the Markov property.

3. Subject CT6 September 2014 Question 9

(i) List the main steps in the Box-Jenkins approach to fitting an ARIMA time series to observed data.

Observations $x_1, x_2, \ldots, x_{200}$ are made from a stationary time series and the following summary statistics are calculated:

$$\sum_{i=1}^{200} x_i = 83.7 \qquad \sum_{i=1}^{200} (x_i - \bar{x})^2 = 35.4 \qquad \sum_{i=2}^{200} (x_i - \bar{x})(x_{i-1} - \bar{x}) = 28.4$$

$$\sum_{i=3}^{200} (x_i - \bar{x})(x_{i-2} - \bar{x}) = 17.1$$

(ii) Calculate the values of the sample auto-covariances $\hat{\gamma}_0$, $\hat{\gamma}_1$ *and* $\hat{\gamma}_2$

(iii) Calculate the first two values of the partial correlation function $\hat{\phi}_1$ and $\hat{\phi}_2$

The following model is proposed to fit the observed data: X_t − μ = a1 (Xt−1 − μ) + et where et is a white noise process with variance σ2.

(iv) Estimate the parameters μ, a1 and σ2 in the proposed model.

After fitting the model in part (iv) the 200 observed residual values ˆ t e were calculated. The number of turning points in the residual series was 110.

(v) Carry out a statistical test at the 95% significance level to test the hypothesis that ˆ t e is generated from a white noise process.

4. Subject CT6 April 2015 Question 7

The following time series model is being used to model monthly data:

$$Y_t = Y_{t-1} + Y_{t-12} - Y_{t-13} + e_t + \beta_1 e_{t-1} + \beta_{12} e_{t-12} + \beta_1 \beta_{12} e_{t-13}$$

where e_t is a white noise process with variance σ^2.

(i) Perform two differencing transformations and show that the result is a moving average process which you may assume to be stationary.

(ii) Explain why this transformation is called seasonal differencing.

(iii) Derive the auto-correlation function of the model generated in part (i).

5. Subject CT6 April 2016 Question 9

Consider the following time series model:

$$Y_t = 1 + 0.6Y_{t-1} + 0.16Y_{t-2} + \varepsilon_t$$

where ε_t is a white noise process with variance σ^2.

(i) Determine whether Y_t is stationary and identify it as an ARMA(p,q) process.

(ii) Calculate $E(Y_t)$.

(iii) Calculate for the first four lags:

- the autocorrelation values $\rho_1, \rho_2, \rho_3, \rho_4$ and
- the partial autocorrelation values $\psi_1, \psi_2, \psi_3, \psi_4$.

6. **Subject CT6 September 2016 Question 9**

In order to model the seasonality of a particular data set an actuary is asked to consider the following model:

$$(1-B^{12})(1-(\alpha+\beta)B+\alpha\beta B^2)X_t=\varepsilon_t$$

where B is the backshift operator and ε_t is a white noise process with variance σ^2.

The actuary intends to apply a seasonal difference $\nabla_s X_t = Y_t$.

(i) Explain why s should be 12 in this case (i.e. $Y_t = X_t - X_{t-12}$).

(ii) Determine the range of values for α and β for which the process will be stationary after applying this seasonal difference.

Assume that after the appropriate seasonal differencing the following sample autocorrelation values for observations of Y_t are $\hat{\rho}_1 = 0$ and $\hat{\rho}_2 = 0.09$.

(iii) Estimate the parameters α and β.

The actuary observes a sequence of observations $x_1, x_2, \ldots, x_T$ of X_t, with $T > 12$.

(iv) Derive the next two forecasted values for next two observations $\hat{x}_{T+1}$ and $\hat{x}_{T+2}$, as a function of the existing observations.

7. **Subject CT6 April 2017 Question 6**

Model A is a stationary AR(1) model, which follows the equation:

(i) State two approaches for estimating the parameters in Model A.

$$y_t = \mu + \alpha y_{t-1} + \varepsilon_t$$

where ε_t is a standard white noise process.

Mary, an actuarial student, wishes to revise Model A such that the error terms εt no longer follow a Normal distribution.

(ii) Explain which of the approaches in part (i) she should now use for parameter estimation.

(iii) Propose a method by which Mary will be able to calculate estimates of the parameters α and σ2, with reference to any relevant equations.
Mary, has now constructed Model B. She has done this by multiplying both sides of the equation above by (1 − cB), where B is the backshift operator, so that Model B follows the equation:

$$y_t(1-cB) = (\alpha y_{t-1} + \varepsilon_t)(1-cB).$$

(iv) Explain why Model A and Model B are identical.

(v) Explain for which values of c Model B is stationary.

8. Subject CT6 September 2017 Question 10

(i) Show that X_t is not stationary.

Let $\Delta X_t = X_t - X_{t-1}$.

(ii) Show that ΔX_t is stationary.

(iii) Determine the autocovariance values of ΔX_t in terms of those of Y_t.

Now assume that Y_t is an MA(1) process, i.e. $Y_t = \varepsilon_t + \beta\varepsilon_{t-1}$

(iv) Set out an equation for ΔX_t in terms of b, β , ε_t and L, the lag operator.

(v) Show that ΔX_t has a variance larger than that of Y_t.

9. Subject CT6 April 2018 Question 9

Consider the following time series model:

(1 – αB)^3*Xt = εt
where B is the backshift operator and et is a standard white noise process with variance s^2.

(i) Determine for which values of a the process Xt is stationary. Now assume that Xt is stationary.

(ii) Calculate the autocorrelation function for the first two lags: r1 and r2, using the Yule-Walker equations.

(iii) State the formulae, in terms of r1 and r2, for the first two values of the partial auto correlation function f1 and f2.

Now assume that a = 1.

(iv) Explain how to fit the parameter of this model, given the time series observations X1, X2, ..., XT

10. Subject CT6 September 2007 Question 10

The time series X_t is assumed to be stationary and to follow an ARMA (2,1) process defined by:

$$X_t = 1 + \frac{8}{15} X_{t-1} - \frac{1}{15} X_{t-2} + Z_t - \frac{1}{7} Z_{t-1}$$

where Z_t are independent $N(0,1)$ random variables.

(i) Determine the roots of the characteristic polynomial, and explain how their values relate to the stationarity of the process. [?

(ii) (a) Find the autocorrelation function for lags 0, 1 and 2.

(b) Derive the autocorrelation at lag k in the form

$$\rho_k = \frac{A}{c^k} + \frac{B}{d^k}$$

[

(iii) Determine the mean and variance of X_t.

11. Subject CT6 September 2018 Question 9

An actuary is modelling a set of data which consists of 100 consecutive observations, $y_1, y_2, \dots, y_{100}$. The data has the following statistics:

$$\bar{y} = \frac{1}{100}\sum_{i=1}^{100} y_i = A = 10.5$$

$$\sum_{i=1}^{100} (y_i - \bar{y})^2 = B = 290$$

$$\sum_{i=2}^{100} (y_{i-1} - \bar{y})(y_i - \bar{y}) = C = 60$$

$$\sum_{i=3}^{100} (y_{i-2} - \bar{y})(y_i - \bar{y}) = D = -240$$

(i) Calculate the values of the sample auto-correlations r_1 and r_2. [3

(ii) Calculate the first two sample partial auto-correlation values $\hat{\phi}_1$ and $\hat{\phi}_2$. [2

The actuary is considering two different models for this data:

Model X: $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$

Model Y: $y_t = b_0 + b_1 y_{t-1} + b_2 y_{t-2} + \varepsilon_t$

where ε_t is a standard white-noise process, with variance σ^2.

(iii) Estimate the parameters (including σ^2) for both Models X and Y, using the method of moments. [10

(iv) Explain whether each of Models X and Y satisfy the Markov property. [3

[Total 18

12.Subject CS2 April 2021 Question 9

Consider the following time series process:
$Y_t = 1 + 0.3 Y_{t-1} + 0.1 Y_{t-2} + e_t$
where e_t is a white noise process with variance σ^2.

(i) Determine whether Y_t is stationary and identify the values of p, d and q for which the process is an ARIMA(p,d,q) process.

Let ρ_k and ϕ_k denote the values at lag k of the autocorrelation and partial autocorrelation functions, respectively.

(ii) Determine the autocorrelation values ρ_1, ρ_2 and ρ_3.

(iii) Determine the partial autocorrelation values ϕ_1, ϕ_2 and ϕ_3.

A sample of the process Y_t is taken in which the sample autocorrelation values are equal to the theoretical values ρ_k .

(iv) Determine the minimum sample size, n, necessary to reject the null hypothesis of a white noise process, under the Ljung and Box 'portmanteau' test using three lags and a 5% significance level.

(v) Discuss the relative merits of using a large or a small number of lags in the Ljung and Box 'portmanteau' test by considering how the value of n in part (iv) would vary if a different number of lags were used or if the sample autocorrelation values were not equal to the theoretical values.

13.Subject CS2 September 2021 Question 7

An Actuary is considering using the following process to model a seasonal data set:

$$(1-B^3)(1-(\alpha+\beta)B+\alpha\beta B^2)X_t = e_t$$

where B is the backwards shift operator and e_t is a white noise process with variance σ^2.

A seasonal difference series is defined as follows:

$$Y_t = X_t - X_{t-3}$$

(i) Express the equation for the original process X_t in terms of the seasonal difference series, Y_t, and the backwards shift operator B. [1

(ii) Determine the range of values of α and β for which the seasonal difference series, Y_t, is stationary. [2

Let γ_k and ρ_k denote the values at lag k of the autocovariance and autocorrelation functions, respectively, of the seasonal difference series, Y_t. The first Yule–Walker equation for Y_t may be written as follows:

$$1-(\alpha+\beta)\rho_1+\alpha\beta\rho_2 = \frac{\sigma^2}{\gamma_0}$$

(iii) Write down the second and third Yule–Walker equations for Y_t in terms of ρ_1 and ρ_2. [2

The Actuary has observed the following sample autocorrelation values for the series Y_t: $\hat{\rho}_1 = 0.5$ and $\hat{\rho}_2 = 0.2$.

(iv) Estimate, using the equations in part (iii), the parameters α and β based on thi information. [5

[**Hint:** let $M = \alpha + \beta$ and $N = \alpha\beta$ and use the formula for finding the roots of a quadratic equation.]

(v) Determine the values of the one-step ahead and two-step ahead forecasts, $\hat{x}_{550}$ and $\hat{x}_{551}$, respectively, based on the parameters estimated in part (iv) and the observed values $x_1, x_2, \ldots, x_{549}$ of X_t. [4

[Total 14

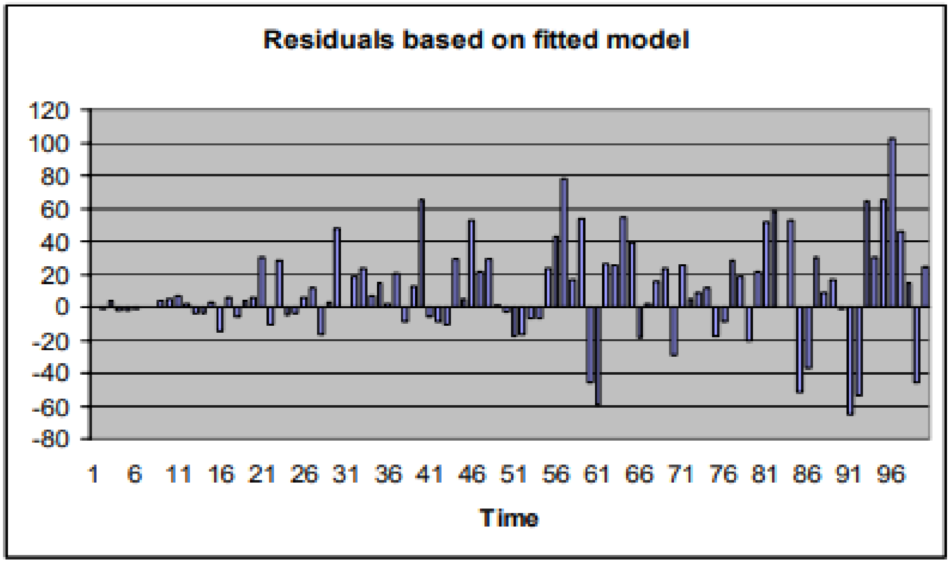
14. Subject CT6 April 2007 Question 3

(i) Explain the concept of cointegrated time series.

(ii) Give two examples of circumstances when it is reasonable to expect that two processes may be cointegrated.

15. Subject CT6 April 2007 Question 8

A modeller has attempted to fit an ARMA(p,q) model to a set of data using the Box-Jenkins methodology. The plot of residuals based on this proposed fit is shown below.



(i) Under the assumptions of the model, the residuals should form a white noise process.

(a) By inspection of the chart, suggest two reasons to suspect that the residuals do not form a white noise process.

(b) Define what is meant by a turning point.

(c) Perform a significance test on the number of turning points in the data above. (There are 100 points in the data and 59 turning points.)

[6]

(ii) On your suggestion, the original fitted model is discarded, and re-parameterised to:

$$X_{n+2} = 5 + 0.9(X_{n+1} - 5) + e_{n+2} + 0.5e_n .$$

Given the following observations:

$X_{99} = 2, \qquad X_{100} = 7$

$\hat{e}_{99} = -0.7, \qquad \hat{e}_{100} = 1.4$

Use the Box-Jenkins methodology to calculate the forward estimates $X_{100}(1), X_{100}(2)$ and $X_{100}(3)$. [4]

[Total 10]

16. Subject CT6 2009 September Question 6

UNIT 3

PRACTICE QUESTIONS

The following data is observed from $n = 500$ realisations from a time series:

$$\sum_{i=1}^{n} x_i = 13153.32 \ , \ \sum_{i=1}^{n}(x_i - \bar{x})^2 = 3153.67 \text{ and } \sum_{i=1}^{n-1}(x_i - \bar{x})(x_{i+1} - \bar{x}) = 2176.03 .$$

(i) Estimate, using the data above, the parameters μ, a_1 and σ from the model

$$X_t - \mu = a_1(X_{t-1} - \mu) + \varepsilon_t$$

where ε_t is a white noise process with variance σ^2. [´

(ii) After fitting the model with the parameters found in (i), it was calculated that the number of turning points of the residuals series $\hat{\varepsilon}_t$ is 280.

Perform a statistical test to check whether there is evidence that $\hat{\varepsilon}_t$ is not generated from a white noise process. [3

17. Subject CT6 September 2010 Question 11

A time series model is specified by

$$Y_t = 2\alpha Y_{t-1} - \alpha^2 Y_{t-2} + e_t$$

where e_t is a white noise process with variance σ^2.

(i) Determine the values of $\alpha$for which the process is stationary.

(ii) Derive the auto-covariances $\gamma 0$ and $\gamma 1$ for this process and find a general recursive expression for γk for $k \geq 2$.

(iii) Show that the auto-covariance function can be written in the form: $\gamma_k = A\alpha^k + kB\alpha^k$

for some values of A, B which you should specify in terms of the constants α and $\sigma 2$.

18. Subject CT6 September 2014 Question 9

UNIT 3

PRACTICE QUESTIONS

(i) List the main steps in the Box-Jenkins approach to fitting an ARIMA time series to observed data. [:

Observations $x_1, x_2, \ldots, x_{200}$ are made from a stationary time series and the followin summary statistics are calculated:

$$\sum_{i=1}^{200} x_i = 83.7 \qquad \sum_{i=1}^{200} (x_i - \bar{x})^2 = 35.4 \qquad \sum_{i=2}^{200} (x_i - \bar{x})(x_{i-1} - \bar{x}) = 28.4$$

$$\sum_{i=3}^{200} (x_i - \bar{x})(x_{i-2} - \bar{x}) = 17.1$$

(ii) Calculate the values of the sample auto-covariances $\hat{\gamma}_0, \hat{\gamma}_1$ and $\hat{\gamma}_2$. [:

(iii) Calculate the first two values of the partial correlation function $\hat{\phi}_1$ and $\hat{\phi}_2$. [:

The following model is proposed to fit the observed data:

$$X_t - \mu = a_1 (X_{t-1} - \mu) + e_t$$

where e_t is a white noise process with variance σ^2.

(iv) Estimate the parameters μ, a_1 and σ^2 in the proposed model.

After fitting the model in part (iv) the 200 observed residual values $\hat{e}_t$ were calculated. The number of turning points in the residual series was 110.

(v) Carry out a statistical test at the 95% significance level to test the hypothesis that $\hat{e}_t$ is generated from a white noise process. [

[Total] 1

19. Subject CT6 April 2015 Question 7

The following time series model is being used to model monthly data:

$$Y_t = Y_{t-1} + Y_{t-12} - Y_{t-13} + e_t + \beta_1 e_{t-1} + \beta_{12} e_{t-12} + \beta_1 \beta_{12} e_{t-13}$$

where e_t is a white noise process with variance σ^2.

(i) Perform two differencing transformations and show that the result is a moving average process which you may assume to be stationary. [3]

(ii) Explain why this transformation is called seasonal differencing. [1]

(iii) Derive the auto-correlation function of the model generated in part (i). [8]

20. Subject CT6 September 2015 Question 11

Consider the following pair of equations:

$$X_t = 0.5X_{t-1} + \beta Y_t + \varepsilon_t^1$$

$$Y_t = 0.5Y_{t-1} + \beta X_t + \varepsilon_t^2$$

where ε_t^1 and ε_t^2 are independent white noise processes.

(i) (a) Show that these equations can be represented as

$$M\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = N\begin{pmatrix} X_{t-1} \\ Y_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \end{pmatrix}$$

where M and N are matrices to be determined.

(b) Determine the values of β for which these equations represent a stationary bivariate time series model.

(ii) Show that the following set of equations represents a VAR(p) (vector auto regressive) process, by specifying the order and the relevant parameters:

$$X_t = \alpha X_{t-1} + \alpha Y_{t-1} + \varepsilon_t^1$$

$$Y_t = \beta X_{t-1} - \beta X_{t-2} + \varepsilon_t^2$$