INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: SY BSc

Subject : Statistical Risk Modelling 1

Chapter: Unit 3 Chapter 2

Chapter Name: Methods of Graduation 2

Today's Agenda

1. Introduction to Graduation
    1. Why Graduation

2. 3 Methods of Graduation

1. Graduation by Parametric Formula

1. Graduation By reference to a Standard table

1. Graduation using Spline functions

1. Comparison of the 3 methods

1 Introduction

- The crude estimates ($\hat{\mu}_x$) will progress erratically from age to age. For example, it may be less than both μ_{x-1} and μ_{x+1}.
- In large part, this is because they have each been estimated independently and hence suffer independent sampling errors.
- For several reasons we would prefer to work with μ_x values that are smooth functions of age.
- Therefore, we graduate or smooth the crude estimates, to produce a set of graduated estimates that do progress smoothly with age.
- We denote these by $\dot{\mu}_x$ (Read as mu circle x)

1.1 Why Graduation

Why do we want smoothed estimates?

- At the heart of our desire to graduate is the intuitive idea that μx should be a smooth function of age. There is some evidence from large investigations to support this, but it is nevertheless an assumption.

- It follows that a crude estimate of μx for any age x also carries information about the values of μ_{x-1}, μ_{x+1} etc. For example, if the force of mortality is smooth and not changing too rapidly, then our estimate of μ_x should not be too far away from estimating μ_{x-1} and μ_{x+1} , as well as being the "best" estimate, in some sense, of . By smoothing, we can make use of the data at adjacent ages to improve the estimate at each age.

- Another way of looking at this is that smoothing reduces the sampling errors at each age

2 3 methods of Graduation

The three methods of graduation are:

1. Graduation by parametric formula
1. Graduation by reference to Standard table
1. Graduation using Spline functions

Most appropriate method will depend on quality of data and purpose of graduation

3 Graduation by Parametric Formula

- The method of graduation **most often used for reasonably large experiences** is to fit a parametric formula to the crude estimates.
- μ_x – modelled **using a mathematical formula**.
- Lower no. of parameters -> Formula is not flexible enough to follow crude rates.
- Higher no. of parameters -> Formula is not smooth enough.

3 Graduation by Parametric Formula

Graduation process:

Steps-

1. Select graduation formula
    1. Gompertz $[\mu_x = Bc^x]$
    2. Makeham $[\mu_x = A + Bc^x]$

[The most recent standard tables produced for use by UK life insurance companies used formulae of the form **μ_x = polynomial(1) + exp{polynomial(2)}** which includes Gompertz and Makeham as special cases.]

2. Determine parameter values
    2. Best fitting values of $\alpha_1, \alpha_2, \ldots \alpha_n$ should be found
    3. Can use MLE, weighted squares estimation, etc
    4. Usually automated i.e., done using statistical packages

3. Calculate graduated rates

4. Test it using statistical tests

4 Graduation by reference to a standard table

A “standard table” means a published life table based upon sufficient data to be regarded as reliable (for appropriate applications). Examples include national life tables based on a country’s entire population (e.g. the English Life Tables)

When and Why do we use this method?

A **standard table will always be based on a well-defined class of lives**, although this does not mean the class of lives will be perfectly homogeneous. If we are given the mortality experience of **a similar group of lives, we might reasonably suppose that it should share some of the characteristics** of the experience underlying the standard table, such as its overall shape. This is useful **if we do not have much data** from the experience in which we are interested

4 Graduation by reference to a standard table

Graduation process:

Steps:

1. Select appropriate standard table:
   i. Class of lives, pattern of mortality should be similar

2. Find simple link to standard table:
   i. Exploit assumed similarity between experience and standard force of mortality rates.

   ii. Example: Simple plots can be made of $\dot{\mu}_x$ vs μ_x^s. (If it is not possible to find a simple relationship, then the supposition that the experiences have similar characteristics should perhaps be reconsidered. It should be remembered that if data are scarce, too close a fit to any suggested relationship is not to be expected, especially at extreme ages)

4 Graduation by reference to a standard table

Graduation process:

Steps:

3. Determine parameter values:
   i. MLE -

   ii. Least squares [the parameter values are found which minimize:

$$\sum_{all\ ages\ x} w_x \left(\hat{\mu}_x - \dot{\mu}_x\right)^2$$

   where the $\{w_x\}$ are suitable weights. Natural weights would be the exposures to risk (E_x^c) at each age, or the inverse of the estimated variance of $\hat{\mu}_x$]

4. Calculate graduated rates:

5. Test.

5 Graduation using Spline functions

An alternative approach to graduation is to use spline functions. These are polynomials of a specified degree which are defined on a piecewise basis across the age range.
The pieces join at knots, where certain continuity conditions for the functions themselves and their derivatives are required

The method may be illustrated using cubic splines, or polynomials of degree 3, which are commonly used. They were used, for example, to graduate English life Table 14 (1980–82). Similar, but more complex methods, have been used to graduate more recent English Life Tables. For example, English Life Table 17 (2010–2012) used a linear combination of basis splines and a penalization term to control the smoothness of the resulting fit.

5 Graduation using Spline functions

Suppose we wish to fit a spline through a set of mortality rates μ_x for ages x with knots at ages x_1 , x_2 , ..., x_n .
The smoothest interpolating spline is the natural cubic spline. This is linear at ages below x_1 and above x_n .

This can be written as: $$\mu_x = \alpha_0 + \alpha_1 x + \sum_{j=1}^{n} \beta_j \phi_j(x) ,$$

where $n \geq 3$ and:

$$\phi_j(x) = \begin{cases} 0 & x < x_j \\ (x - x_j)^3 & x \geq x_j \end{cases} .$$

This leads to the following form for the natural cubic over the whole age range

$$\mu_x = \alpha_0 + \alpha_1 x + \sum_{j=1}^{n-2} \beta_j \Phi_j(x) ,$$

5 Graduation using Spline functions

Graduation process:

Steps:

1. Make decisions about knots (age groups)
   - not necessary to be equally spaced

2. Preliminary calculations – calculate $\Phi_j(x)$

2. Estimate parameter values
   - Values of $\alpha_0, \alpha_1, \beta_1 \beta_2 \beta_3 \ldots\ldots \beta_n$ are determined using weighted least squares method.

2. Calculate graduated rates

2. Test

Question

CT4 September 2012 Q8

(i) Describe a situation when graduation of raw mortality data using a parametric formula might be appropriate and explain why. [2]

(i) (a) State another method of graduation.
(b) Suggest a situation in which its use may be appropriate. [1]

Solution

(i) When preparing standard tables OR when graduating data from a large industrywide scheme, or a national population because there will be lots of data available.

(i) (a) Graduation with reference to a standard table

(b) Graduation with reference to a standard table is useful if data are scanty and a suitable standard table exists (e.g., for female pensioners from a small scheme).

6 Comparison of the 3 methods

First, we note that the three methods of graduation described above by no means cover all possible methods. We take parametric formula graduation to be an example of approaches used with reasonably large data sets, and the other two to be examples of methods used with smaller data sets.

6 Comparison of the 3 methods

Parametric Formula

- It is a natural extension of the simple probabilistic models for single years of age, parameterized by μ_x.
- The graduation will inherit its smoothness from the smoothness of the function in the parametric formula. In general, formulae with a small number of parameters will produce an acceptably smooth graduation.
- The approach is very well-suited to the production of standard tables from large amounts of data.
- It can, however, be very difficult to find a suitable curve that fits an experience well at all ages. Partly this is because of the different features that predominate at different ages. Partly it may be because cross-sectional studies mix up different generations at different ages. A very likely reason is that there is still a good deal of heterogeneity in all mortality studies, even if we classify the data by age, sex, policy type, calendar period and so on. Also, extrapolation to age ranges where data is not available may be difficult.

6 Comparison of the 3 methods

Standard Table

- It can be used to fit relatively small data sets where a suitable standard table exists.
- Provided a simple function is chosen and the standard table is smooth to begin with, a smooth graduation should result.
- The collateral information obtained from the standard table can be particularly useful in deciding the shape of the graduation at the extreme ages, where there might be little or no data.
- The method is not suitable for the preparation of standard tables based on large amounts of data.
- The choice of standard table is important; choosing an inappropriate table could impart the wrong shape to the entire graduation.
- It is not always easy to choose an appropriate standard table.

6 Comparison of the 3 methods

Spline Functions

- Provided the number of knots is small, the graduation will usually be smooth.
- Alternative graduations can be tried by varying the number and position of the knots.
- The method is suitable for quite small experiences as well as very large experiences (such as national populations). It can also be used to produce standard tables. It is, however, not suitable for very small experiences with scanty data at many ages.
- It is not easy to choose the knots, and experiments with different numbers and locations are likely to be needed.
- Care is required at extreme ages where data can be scanty. With a natural cubic spline, the form of the curve above the highest knot is linear. Therefore, the fit at the oldest ages can be quite poor and need adjustment.

Question

CT4 September 2007 Q5

(i) Explain why crude mortality rates are graduated before being used for financial calculations. [3]

(i) List two methods of graduating a set of crude mortality rates and state, for each method:
(a) under what circumstances it should be used; and
(b) how smoothness is ensured [4]
[Total 7]

Solution

(i) We assume that mortality rates progress smoothly with age. Therefore, a crude estimate at age x carries information about the rates at adjacent ages, and graduation allows us to use this fact to “improve” the estimate at age x by smoothing.

This reduces the sampling errors at each age. It is desirable that financial quantities progress smoothly with age, as irregularities are hard to justify to clients.

Solution

(ii)Two methods:

By parametric formula:
Should be used for large experiences, especially if the aim is to produce a standard table;
Depends on a suitable formula being found which fits the data well.
Provided the number of parameters is small, the resulting curve should be smooth.

With reference to a standard table:
Should be used if a standard table for a class of lives similar to the experience is available, and the experience we are interested in does not provide much data.
The standard table will be smooth and;
provided the function linking the graduated rates to the rates in the standard table is simple, this smoothness will be “transferred to the graduated rates”.