PUSASQF402
Statistical & Risk Modelling - 2

Time: 2 hours
Total Marks: 60 marks

Note:

1) **The candidate has option to either attempt question 4A or question 4B. Rest all questions are mandatory.**
2) **Numbers to the right indicate full marks.**
3) **The candidates will be provided with the formula sheet and graphpapers (if required) for the examination.**
4) **The candidates should only write the option alphabet for answersin case of the MCQ based questions.**
5) **Use of approved scientific calculator is allowed.**

Q1A **5 Marks**

Explain the following terms:

a) Strict stationarity [2]
b) Weak stationarity [1]
c) Independent increments [1]
d) Filtration [1]

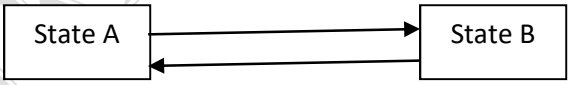
Q1B **5 Marks**

a) Define the irreducibility and periodicity of a Markov chain. [2]

b) State whether the following chains are [3]

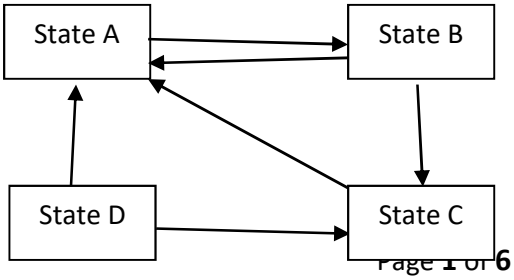
i) irreducible
ii) periodic, mentioning the period as well

1)



2)

3)



Q1C **5 Marks**

a) Define a Markov jump process and state the main difference between time homogeneous and time inhomogeneous Markov jump processes. [3]

b) What are the principal difficulties in modeling using a time inhomogeneous Markov jump process? [2]

Q2A. **[5 Marks]**

An investigation was carried out into the relationship between sickness and mortality in an historical population of working class men. The investigation used a three-state model with the states:

1 Healthy
2 Sick
3 Dead

Let the probability that a person in state i at time x will be in state j at time x+t be ${}_tp^{ij}_x$.

Let the transition intensity at time x+t between any two states i and j be μ^{ij}_{x+t} .

(i) Show from first principles that

$$\frac{\partial}{\partial t}{}_tp^{23}_x = {}_tp^{21}_x\mu^{13}_{x+t} + {}_tp^{22}_x\mu^{23}_{x+t}$$ [5]

Q2B **[5 Marks]**

The two cricket teams in a particular country are called RCB and CSK and there is intense rivalry between them. A researcher has collected the following history on the results of the last 20 matches between the teams from the earliest to the most recent, where:

R indicates a win for RCB;
C indicates a win for CSK;
D indicates a draw.

RCCDDRCDCRRDRDCCRDCC

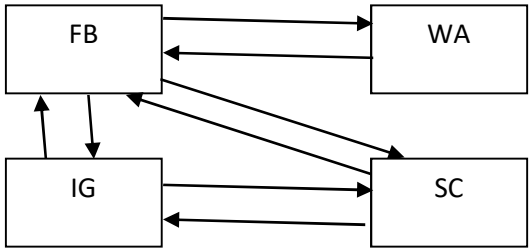
The researcher has assumed that the probability of each result for the next match depends only on the most recent result. He therefore decides to fit a Markov chain to this data.

(i) Estimate the transition probabilities for the Markov chain. [2]

(ii) Estimate the probability that RCB will win at least two of the next three matches against CSK. [3]

Q2C **[5 Marks]**

A simplified model of the internet consists of the following websites with links between the websites as shown in the diagram below.



An internet user is assumed to browse by randomly clicking any of the links on the website he is on with equal probability.

(i) Calculate the transition matrix for the Markov chain representing which website the internet user is on. [2]

(ii) Calculate, of the total number of visits, what proportion are made to each website in the long term. [3]

Q3A **5 Marks**

Steve Jobs' biography is very popular these days at a college library. Keeping in mind the demand, the librarian wants to understand students' reading patterns and in particular how these influence the availability of this book at the library.

The librarian had read a book on Markov Jump Processes and realized these can be applied for his analysis. He decides to use a two-state Markov Process with the states being reflecting the status of the book - borrowed (B) or available (A).
The fortnightly transition rates between these states are given below:

$$A\ B\ \begin{bmatrix} -0.9 & 0.9 \\ 0.1 & -0.1 \end{bmatrix}$$

The probabilities of the book being either borrowed or available at time t are denoted by $P_B(t)$ and $P_A(t)$
a) Write down Kolmogorov's forward equation for $P_B'(t)$.
b) Solve the equation in part (a) to obtain a formula for the probability that the book is borrowed at time t, given that it was available at time 0.

5 Marks

Q3B By using an indicator function or otherwise, calculate the expected period of time the book is available over the period $[0, t]$ and hence calculate the expected number of days the book is available over a fortnight (15 days).

Q3C **5 Marks**

A recording instrument is set up to observe a continuous time process, and stores the results for the most recent 500 transitions. The data collected are as follows:

State	Total time Number of transitions to i spent in state i (hours)	No. of transitions to		
		State A	State B	State C
A	50	NA	80	60
B	125	100	NA	75
C	250	160	25	NA

It is proposed to fit a Markov jump model using the data.

(i) (a) Estimate the parameters of the model.

(b) Write down the estimated generator matrix of the model. [4]

(ii) Specify the distribution of the number of transitions from state i to state j, given the number of transitions out of state i. [1]

Total [5]

Q4A **15 Marks**

A shop has four refrigerator storages in which specialized energy drinks are held for sale. The shop can stock at most four drinks at any one time because:

- one storage can only accommodate one drink due to its specialized packing
- if drinks are kept outside the box, they will lose its potency

The number of drinks sold by the shop each day is a random variable with the following distribution:

Number of Drinks Potentially Sold in Day (if stock is sufficient)	Probability
None	0.3
One	0.5
Two	0.2

If the shop has no drinks in stock at the end of a day, the owner contacts his drink supplier to order four more drinks. The drinks are delivered the following morning before the shop opens.

(i) Write down the transition matrix for the number of drinks in stock when the shop opens in a morning, given the number in stock when the shop opened the previous day. [2]

(ii) Calculate the stationary distribution for the number of drinks in stock when the shop opens, using your transition matrix in part (i). [4]

(iii) Calculate the expected long term average number of restocking orders placed by the shop owner per trading day. [2]

If a customer arrives intending to purchase a drink, and there is none in stock, the sale is lost to a rival shop.

(iv) Calculate the expected long term number of sales lost per trading day. [2]

The owner is unhappy about losing these sales as there is a profit on each sale. He therefore considers changing his restocking approach to place an order before he has run out of drinks.

(v) Evaluate the expected number of restocking orders, and number of lost sales per trading day, if the owner decides to restock if there are fewer than two drinks remaining in stock at the end of the day. [5]

OR

Q4B **15 Marks**

A) Outside an apartment block there is a small car park with three parking spaces. A prospective purchaser of an apartment in the block is concerned about how often he would return in his car to find that there was no empty parking space available. He decides to model the number of parking spaces free at any time using a time homogeneous Markov Jump Process where:

• The probability that a car will arrive seeking a parking space in a short interval dt is $A.dt + o(dt)$.
• For each car which is currently parked, the probability that its owner drives the car away in a short interval dt is $B.dt + o(dt)$.
where $A, B > 0$.

(i) Specify the state space for the above process. [1]

(ii) Draw a transition graph of the process. [2]

(iii) Write down the generator matrix for the process. [2]

(iv) Derive the probability that, given all the parking spaces are full, they will remain full for at least the next three hours. [2]

(v) Specify the transition matrix for the jump chain associated with this process. [2]

Assuming $A = 0.2$ and $B = 0.15$, suppose there are currently two empty parking spaces.

(vi) Determine the probability that all the spaces become full before any cars are driven away. [1]

(vii) Derive the probability that the car park becomes full before the car park becomes empty. [3]

(viii) Comment on the prospective purchaser's assumptions regarding the arrival and departure of cars. [2]