Lecture

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: MSc
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Chapter : Unit 3 Chapter 1

Chapter Name: Risk and Ruin Theoíy

Agenda

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1.1 Introduction

- We consider models for analyzing the surplus of an insurance portfolio.
- Suppose an insurance business begins with a start-up capital, called the initial surplus.
- The insurance company receives premium payments and pays claim losses.
- The premium payments and claim amounts are assumed to be coming in at a constant rate.
- When there are claims, losses are paid out to policy holders. Unlike the constant premium payments, losses are random and uncertain, in both timing and amount.
- The net surplus through time is the excess of the initial capital and aggregate premiums received over the losses paid out. The insurance business is in ruin if the surplus falls to or below zero.
- The main purpose of this chapter is to consider the probability of ruin as a function of time, the initial surplus and the claim distribution.

1.2 Ruin Theory

- To prevent ruin, people purchase insurance.
    - Capital Reserves need to be calculated to cover these risks.

- If the risk is realized, and the insurer cannot pay then they're "ruined"
    - They pay insurer a premium.
    - And the insurer takes on the risk.
        i. If reserves are too low, the probability of ruin will be high
        ii. If reserves are too high, shareholders will get poor return.
        Will withdraw their capital and the insurer will close down

- Benefits of Ruin Theory Model
    - Helps us calculate how much initial capital we need
    - Helps us see the impact of various premium pricing strategies will have of ruin
        i. Higher premiums could reduce the probability of ruin
        ii. But higher premiums might result in less policyholders!

1.3 Aggregate Claims Process

- In the actuarial literature, the word 'risk' is often used instead of the phrase 'portfolio of policies'. In this unit both terms will be used, so that by a 'risk' will be meant either a single policy or a collection of policies. This section will focus on claims generated by a portfolio over successive time periods. Some notation is needed.

- $N(t)$ the number of claims generated by the portfolio in the time interval $[0, t]$, for all $t \geq 0$
  Xi the amount of the i-th claim, i = 1, 2, 3, ...

- $S(t)$ the aggregate claims in the time interval $[0, t]$, for all $t \geq 0$.

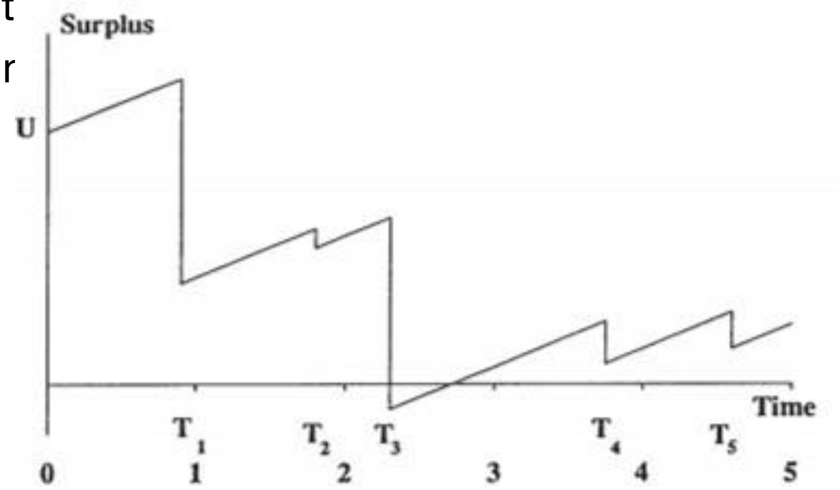
- $\{Xi\}$ is a sequence of random variables. $\{N(t)\}_{t\geq0}$ *and* $\{S(t)\}_{t\geq0}$ are both families of random variables, one for each time $t \geq 0$; in other words $\{N(t)\}_{t\geq0}$ *and* $\{S(t)\}_{t\geq0}$ are stochastic processes.

- It can be seen that: $S(t) = \sum_{i=1}^{N(t)} X_i$ with the understanding that $S(t)$ is zero if $N(t)$ is zero. The stochastic process $\{S(t)\}_{t\geq0}$ as defined above is known as the aggregate claims process for the risk.

1.4 Surplus Process

- Suppose that at time 0 the insurer has an amount of money set aside for this portfolio.
- This amount of money is called the initial surplus and is denoted *U*. It will always be assumed that $U \geq 0$.
- The insurer needs this initial surplus because the future premium income on its own may not be sufficient
  to cover the future claims. Here we are ignoring expenses.
- The insurer's surplus at any future time *t* (> 0) is a random variable since its value depends on the claims experience up to t
- The insurer's surplus at tir … on for *U*(*t*) can be written:

$$U(t) = U + ct - S(t)$$



1.5 Poisson Process

The Poisson process is an example of a counting process. Here the number of claims arising from a risk is of interest. Since the number of claims is being counted over time, the claim number process $\{N(t)\}t \geq 0$ must satisfy the following conditions:

i. $N(0) = 0$, i.e. there are no claims at time 0

ii. for any $t > 0$, $N(t)$ must be integer valued

iii. when $s < t$, $N(s) \leq N(t)$, i.e. the number of claims over time is non-decreasing

iv. when $s < t$, $N(t) - N(s)$ represents the number of claims occurring in the time interval (s, t).

The claim number process $\{N(t)\}_{t \geq 0}$ is defined to be a Poisson process with parameter l if the following conditions are satisfied:

i. $N(0) = 0$, and $N(s) \leq N(t)$ when $s < t$

ii. $P(N(t + h) = r \mid N(t) = r) = 1 - \lambda h + o(h)$

1.6 Compound Poisson Process

The Poisson process for the number of claims will be combined with a claim amount distribution to give a compound Poisson process for the aggregate claims process.

The following three important assumptions are made:

i. The random variables $\{Xi\}_{i=1}^{\infty}$ are independent and identically distributed

ii. The random variables $\{Xi\}_{i=1}^{\infty}$ are independent of $N(t)$ for all $t \geq 0$

iii. The stochastic process $\{N(t)\}_{t\geq 0}$ is a Poisson process whose parameter is denoted λ.

1.7 Probability of Ruin

	Continuous time	Discrete time
Infinite time	• Compound Poisson Process • Brownian Motion	• Poisson Process • Counting Process
Finite time	• General Random Walk • Time Series	• No Claim Discount • Random Walk

1.8 Effect of changing parameter values on ruin probabilities

1) Ψ(U,t) as a function of t

Ψ(U,t) is the probability of ruin at some point of time between 0 and t.
This should increase with time t since longer the time period, more the chance there is of ruin.

2) Ψ(U,t) as a function of Initial Surplus

This should be a decreasing function. The bigger the initial surplus, the less chance there should be of ruin.

3) Ψ(U,t) as a function of Premium Loading

This should be a decreasing function. If everything else remains unchanged, then increasing the premium income will reduce the probability of ruin.

1.8 Effect of changing parameter values on ruin probabilities

4) $\Psi(U,t)$ as a function of Poisson Parameter

An increase in the value of Poisson parameter λ will not affect the probability of ultimate ruin since the aggregate claims E(S) = $\lambda\mathrm{E}(\mathrm{X})$, the variance and premium rate all increase proportionately in line with λ.

5) $\Psi(U,t)$ as a function of Variance

This should be an increasing function. The increase in variance of individual claim amounts will increase the probability of ruin as it will increase the uncertainty associated with the aggregate claims process without any corresponding increase in premium.

6) $\Psi(U,t)$ as a function of Expected individual claim amount

This should be a increasing function. Greater the claim amount more the chance of ruin.

Question

A general insurance company is planning to set up a new class of travel insurance. It plans to start the business with £2 million and expects claims to occur according to a Poisson process with parameter 50. Individual claims are thought to have a gamma distribution with parameters $\alpha = 150$ and $\lambda = \frac{1}{4}$. A premium loading factor of 30% is applied.

Explain how each the following changes to the company's model will affect the probability of *ultimate* ruin:

(i) A 28% premium loading factor is applied instead.

(ii) Individual claims are found to have a gamma distribution with parameters $\alpha = 150$ and $\lambda = \frac{1}{2}$.

(iii) The Poisson parameter is now believed to be 60.

Question

An insurer plans to issue 5,000 one-year policies at the start of a year. For each policy the annual aggregate claims have a compound negative binomial distribution; the negative binomial parameters are $k = 0.5$ and $p = 0.5$, and individual claim amounts, in pounds, have a lognormal distribution with parameters $\mu = 5.04$ and $\sigma = 1.15$.

The premium for each policy is £160 and is payable at the start of the year. Claims are assumed to be paid at the mid-point of the year. Calculate the minimum annual rate of interest the insurer must earn throughout the year if the accumulation to the end of the year of premiums minus claims is to exceed £52,500 with probability 90%. You may assume that the distribution of total aggregate claims in the year may be approximated by a normal distribution.

1.9 Adjustment Coefficient

The adjustment coefficient, denoted R, is defined to be the unique positive root of:

$$\lambda M_X(r) - \lambda - cr = 0$$

So, R is given by:

$$\lambda M_X(R) = \lambda + cR$$

Question

A Poisson claims process has security loading $\theta = 2/5$ and claim size density function:

$f(x) = \frac{3}{2}\, e^{-3x} + \frac{7}{2}\, e^{-7x}\;, x > 0$

(i) Derive the moment generating function (MGF) for the claim size distribution, and state the values of *t* for which it is valid.

(ii) Calculate the value of the adjustment coefficient.

1.10 Lundberg's Inequality

Lundberg's inequality states that:
$\Psi(U) \leq \exp(-RU)$

where U is the insurer's initial surplus and $\Psi(U)$ is the probability of ultimate ruin.

R is a parameter associated with a surplus process known as the adjustment coefficient. Its value depends upon the distribution of aggregate claims and on the rate of premium income.

R can be interpreted as measuring risk. The larger the value of R, the smaller the upper bound for $\Psi(U)$ will be.

Hence, $\Psi(U)$ would be expected to decrease as R increases. R is a function of the parameters that affect the probability of ruin, and R's behaviour as a function of these parameters can be observed.

1.11 Effect of Reinsurance on ruin probabilities

Without Reinsurance

Net Surplus:

$$U(t) = U + ct - S(t)$$

Where c is premium

The adjustment coefficient, denoted R, is defined to be the unique positive root of:

$$\lambda M_X(R) - \lambda - cR = 0$$

With Reinsurance

Net Surplus:

$$U_{net}(t) = U + c_{net}t - S_I(t)$$

Where c= $(1+\theta)E[S_I(t)] - (1+\lambda)E_I[S(t)]$

The adjustment coefficient, denoted R, is defined to be the unique positive root of:

$$\lambda M_Y(R) - \lambda - c_{net}R = 0$$

Question

Claims occur on a portfolio of insurance policies according to a Poisson process with Poisson parameter λ . Claim amounts, $X1$,$X2$,..., are assumed to be identically distributed with moment generating function $MX(t)$. The insurer calculates premiums using a loading factor θ (> 0) . The insurer's adjustment coefficient, R , is defined to be the smallest positive root of the equation:

$\lambda + cr = \lambda MX(r)$

where c is the insurer's premium income rate.

(i) Using the above equation for R , or otherwise, show that, provided R is small, an approximation to R is $\hat{R}$, where:

$$\hat{R} = \frac{2(\frac{c}{\lambda} - \mu)}{\sigma^2 + \mu^2}$$

where $\mu = E[Xi]$ and $\sigma^2 = \text{var}[Xi]$.

Question

(ii) Describe how the adjustment coefficient can be used to assess reinsurance arrangements on the basis of security.

(iii) The Poisson parameter, λ , for this portfolio is 20 and all individual claims are for a fixed amount of £5,000. The insurer's premium loading factor, θ , is 0.15 and proportional reinsurance can be purchased from a reinsurer who calculates premiums using a loading factor of 0.25.

Calculate the maximum proportion of each claim that could be reinsured so that the insurer's security, measured by $\hat{R}$, is greater than the insurer's security without reinsurance.

Question

(i) Show that the adjustment coefficient for a compound Poisson claims process satisfies the inequality:

$$r < \frac{2\left[\frac{c}{\lambda} - E(X)\right]}{E(X^2)}$$

and define what each of the symbols represents.

(ii) An insurer considers that claims of a certain type occur in accordance with a compound Poisson process. The claim frequency for the whole portfolio is 100 per annum and individual claims have an exponential distribution with a mean of £8,000.

(a) Calculate the adjustment coefficient if the total premium rate for the portfolio is £1,000,000 per annum.

(b) Verify that the value calculated in (ii)(a) satisfies the inequality in (i).

(c) The insurer decides to take out excess of loss reinsurance for this portfolio. The reinsurer has agreed to pay the excess of any individual claim above £20,000 in return for an annual premium of £80,000. Calculate the adjustment coefficient for the direct insurer when the reinsurance is in operation.

(d) Estimate the direct insurer's probability of ultimate ruin with and without the reinsurance arrangement, assuming that the initial surplus is £20,000 and that future premiums remain at the same level.