Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc

Subject : Statistical & Risk Modelling - 2

Chapter: Unit 1 Chapter 2

Chapter Name: Application of compound distribution in risk modelling

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6. Parameter variability and uncertainty
7. Approximation of Aggregate Distribution

1.0 Introduction

CLAIMS

FREQUENCY AMOUNT

Both are random and
have a unique distribution

Therefore, to model aggregate claims, we deal with two distributions at the same time.

By combining these two distributions, we define a unique aggregate distribution known as Compound Distribution.

2.0 Models for short-term insurance contracts

Many forms of non-life insurance can be regarded as short-term contracts, for example motor insurance.

Some forms of life insurance also fall into this category, for example group life and one-year term assurance policies.

A short-term insurance contract can be defined as having the following attributes:

- The policy lasts for a fixed, and relatively short, period of time, typically one year.
- The insurance company receives from the policyholder(s) a premium.
- In return, the insurer pays claims that arise during the term of the policy.

2.1 Features

- In short term insurance, premiums are calculated in a way that they cover the expected claims for the entire policy duration (usually 1 year).
- This approach defers from approach used in traditional life insurance where we calculate premium such that they meet the cost of claims in earlier years as well as reserves for later years.
- Accordingly, to compute premiums for a short term insurance contract, we calculate the expected aggregate claims arising from those policies throughout the policy duration.
- Since the claims are random in nature (mentioned earlier) we compute an aggregate claims distribution as follows:

$$S = \sum_{i=1}^{N} X_i$$

3.0 Collective Risk Model

Recall that S is represented as the sum of N random variables Xi , where Xi denotes the amount of the i-th claim. Thus:

S = $X_1 + X_2 + \ldots + X_N$

and S = 0 if N = 0.

S = aggregate/ collective claim amount

X_i = individual claim amount (Discrete/ continuous)

N = number of claims for the policy duration (Discrete)

Then S ~Compound distribution.

Example: If N~ Bin( ) and X~ a given distribution, then

S~ Compound Binomial distribution

4.0 Moments of S

To calculate the moments of S, conditional expectation results are used, conditioning on the number of claims, N. To find E[S], apply the identity:

$E[S] = E[E[S|N]]$.

Formula has a very natural interpretation. It says that the expected aggregate claim amount is the product of the expected number of claims and the expected individual claim amount.

4.0 Moments of S

To find an expression for var[S], apply the identity:

var[S] = E[var[S|N]] + var[E[S|N]]

var(S|N) can be found by using the fact that individual claim amounts are independent.

4.0 Moments of S

To find an expression for MGF of S,

5.0 Individual Risk Model

Claims arising from each individual risk/ policy.

S = $Y_1 + Y_2 + \ldots + Y_N = \sum_{i=1}^{n} Y_j$

$N_1 = 0 \; or \; 1 \quad , N_2 = 0 \; or \; 1$

Y_j = Claim amount under jth risk

j = 1, 2, 3, … n

n = number of risk/ policies

5.0 Individual Risk Model

Assumptions:

Number of claims under ‘j’ th policy is either ‘0’ or ‘1’ (maximum number of claims from each policy is 1)

Probability of claim under j th risk is q_i

$$N_j \sim Bin(1, q_j)$$

If $N_j = 0$, $Y_j = 0$

Otherwise,
$N_j = 1$, $Y_j = X_j$

If a claim occurs under j th risk, claim amount is denoted by a random variable x_j

$X_j \sim$ distribution with mean μ_j *and var* σ_j^2

5.0 Individual Risk Model

Y_j ~ Compound distribution with individual claim amount X_j and

N_j ~Bin$(1, q_j)$, $E(N_j) = q_j$

V(N_j) = $q_j(1 - q_j)$

$E(Y_j) = E(N_j)E(X_j) \;\; = q_j N_j$

Var$(Y_j) = E(X_j)^2 V(N) + E(N).V(X)$

$= \mu_j^2 q_j(1 - q_j) + \sigma_j^2 q_j$

Assumptions (For the model):

1. Fixed number of risks/ policies.
2. Number of risks do not change overtime period of insurance cover.
3. Risks are independent of each other.
4. Claim amounts under risk need not be necessary identical.

$$S_j = \sum_{i=1}^{n} Y_j$$

5.1 Compound Poisson distribution

$N \sim Poi(\lambda)$

S then has a compound Poisson distribution with parameter λ, and $F(x)$ is the CDF of the individual claim amount random variable. The results required for this distribution for N are:

$E[N] = var[N] = \lambda$

$M_N(t) = \exp\{\lambda(e^t - 1)\}$

$E[S] = \lambda m1$

$var[S] = \lambda m2$

$M_S(t) = \exp\{\lambda(M_X(t) - 1)\}$

5.2 Compound Binomial distribution

Under certain circumstances, the binomial distribution is a natural choice for *N*.

$N \sim Bin(n, p)$

The key results for this distribution are:

$E[N] = np$

$var[N] = np(1 - p)$

$M_N(t) = (pe^t + 1 - p)^n$

When *N* has a binomial distribution, *S* has a compound binomial distribution. One important point about choosing the binomial distribution for *N* is that there is an upper limit, *n*, to the number of claims.

5.2 Compound Binomial distribution

Accordingly, the moments of S are:

E[S] = npm_1

Var[S] = np$(m_2 - m_1^2) + np(1-p)m_1^2$

$= npm_2 - np^2 m_1^2$

M_S(t) = $(pM_X(t) + 1 - p)^n$

Use of Compound Binomial Distribution:

A group life insurance policy covering *n* lives with the same mortality rate, the distribution of the number of deaths in a year is binomial.

5.3 Compound Negative Binomial distribution

The final choice of distribution for N is the negative binomial distribution, which has probability function:

P(N = n) = $\frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)} p^k q^n$ for n = 0, 1, 2, ...

The parameters of the distribution are k (> 0) and p, where p + q = 1 and 0 < p < 1. This distribution is denoted by NB(k, p). When N ~ NB(k, p):

E[N] = kq / p

var[N] = kq / p^2

M_N(t) = $p^k(1 - qe^t)^{-k}$

5.3 Compound Negative Binomial distribution

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$var[N] = kq / p^2$

$M_N(t) = p^k(1 - qe^t)^{-k}$

5.3 Compound Negative Binomial distribution

Accordingly, the moments of S are:

E[S] = $\frac{kq}{p} m_1$

Var[S] = $\frac{kq}{p}(m_2 - m_1^2) + \frac{kq^2}{p^2} m_1^2 = \frac{kq}{p} m_2 + \frac{kq^2}{p^2} m_1^2$

M_S(t) = p^k / (1 − $qM_X(t))^k$

6.0 Parameter variability and uncertainty

Introduction

In some cases, parameters of the assume distributions may also be uncertain.
Example: Poisson parameter for claims arising out of a motor insurance policy for a good driver and a bad driver.

In such cases, the resultant compound distribution for aggregate claims will need to take into account this parameter uncertainty, by usually modelling the parameter using a known distribution.

6.0 Parameter variability and uncertainty

Variability in a heterogeneous portfolio

Example: Motor insurance policies for drivers
Good drivers – N~ Poi(0.2)
Bad drivers – N~ Poi(0.4)

i) Find E(Y_i) $and\ var(Y_i)$ for a randomly chosen policy.
ii) Find E(S) and Var(S) for the entire portfolio where S= $\sum Y_i$.

6.0 Parameter variability and uncertainty

Variability in a homogeneous portfolio

Example: Property damage insurance policy is in coastal region.
In good weather year – N~ Poi(0.2)
In bad weather year – N~ Poi(0.4)

i) Find E(Y_i) and $var(Y_i)$ for a randomly chosen policy.
ii) Find E(S) and Var(S) for the entire portfolio where S= $\sum Y_i$.

7.0 Approximation of Aggregate Distribution

CT6 April 2008 Q10

A bicycle wheel manufacturer claims that its products are virtually indestructible in accidents and therefore offers a guarantee to purchasers of pairs of its wheels. There are 250 bicycles covered, each of which has a probability p of being involved in an accident (independently). Despite the manufacturer's publicity, if a bicycle is involved in an accident, there is in fact a probability of 0.1 for each wheel (independently) that the wheel will need to be replaced at a cost of £100. Let S denote the total cost of replacement wheels in a year. (i) Show that the moment generating function of S is given by

$$M_S(t) = \left[\frac{pe^{200t} + 18pe^{100t} + 81p}{100} + 1 - p\right]^{250}$$

(ii) Show that $E(S) = 5{,}000\,p$ and $Var(S) = 550{,}000\,p - 100{,}000\,p^2$

Suppose instead that the manufacturer models the cost of replacement wheels as a random variable T based on a portfolio of 500 wheels, each of which (independently) has a probability of 0.1p of requiring replacement.

(iii) Derive expressions for $E(T)$ and Var(T) in terms of p.

(iv) Suppose $p = 0.05$.

(a) Calculate the mean and variance of S and T.

(b) Calculate the probabilities that S and T exceed £500.

(b) Comment on the differences.

7.0 Approximation of Aggregate Distribution

CT6 April 2015 Q8

The number of claims, *N*, in a given year on a particular type of insurance policy is given by:

$$P(N = n) = 0.8 \times 0.2^n \quad n = 0, 1, 2, \ldots$$

Individual claim amounts are independent from claim to claim and follow a Pareto distribution with parameters α = 5 and λ = 1,000.

(i) Calculate the mean and variance of the aggregate annual claims per policy.

(ii) Calculate the probability that aggregate annual claims exceed 400 using:

(a) a Normal approximation.

(b) a Lognormal approximation.

(iii) Explain which approximation in part (ii) you believe is more reliable.

7.0 Approximation of Aggregate Distribution

CT6 April 2012 Q4

Claims on a particular type of insurance policy follow a compound Poisson process with annual claim rate per policy 0.2. Individual claim amounts are exponentially distributed with mean 100. In addition, for a given claim there is a probability of 30% that an extra claim handling expense of 30 is incurred (independently of the claim size). The insurer charges an annual premium of 35 per policy.

Use a normal approximation to estimate how many policies the insurer must sell so that the insurer has a 95% probability of making a profit on the portfolio in the year.

7.0 Approximation of Aggregate Distribution

CT6 Sept 13 Q7

An insurance company offers dental insurance to the employees of a small firm. The annual number of claims follows a Poisson process with rate 20. Individual loss amounts follow an exponential distribution with mean 100. In order to increase the take-up rate, the insurance company has guaranteed to pay a minimum amount of £50 per qualifying claim. Let S be the total claim amount on the portfolio for a given year.
(i) Show that the mean and variance of S are 2,213.06 and 413,918.40 respectively.

[You may use without proof the result that if $I_n = \int_M^\infty y^n \lambda e^{-\lambda y} dy$ then $I_n = M^n e^{-\lambda M} + \frac{n}{\lambda} I_{n-1}$]

(ii) (a) Fit a log-normal distribution for S using the method of moments.

(b) Estimate the probability that S is greater than 4,000.

Sarah, the insurance company's actuary, has instead approximated S by a Normal distribution.
(iii) Explain, without performing any further calculations, whether the probability that she calculates that S exceeds 4,000 will be greater or smaller than the calculation in part (ii).