INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY BSc

Subject : Calculus

Chapter: Unit 1 – Chapter 2

Chapter Name: Limits & continuity

Today's Agenda

1. What is limits
   1. Why to study limits?
   2. Application of limits
2. Definition of Limit
3. One sided limits
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1 What is Limits?

A limit tells us the value that a function approaches as that function's inputs get closer and closer to some number.

The concept of limits is fundamental to bridging the gap between elementary mathematics and calculus. In fact limits provide the theoretic background on which the main tool of calculus, the derivative is built.

In formulas, a limit of a function is usually written as: $\lim_{x \to c} f(x) = L,$

It is read as "the limit of *f* of *x* as *x* approaches *c* equals *L*".

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1.1 Why to study limits ?

- Limits express the concepts of infinite small and infinite large quantities in mathematical terms.

- Limits are used to determine the values of indeterminate

- The concept of a limit of a function is essential to the study of calculus. It is used in defining some of the more important concepts in calculus: continuity, the derivative of a function, and the definite integral of a function.

- The limit of a function $f(x)$ describes the behavior of the function close to a particular x value. It does not necessarily give the value of the function at x.

1.1 Application of limits in daily life

A good example is continuous compounding of interest. Suppose that the money in your bank account has an annual interest rate of r and it is compounded n times per year. If you initially had M0 dollars in your account then after t years your money has grown to:

$$M_0\left(1+\frac{r}{n}\right)^{nt}.$$

In continuous compounding your money is compounded every infinitesimal time step. This is a little non-rigorous but you can think about it as taking the number of times per year your account is compounded to infinity:

$$\lim_{N\to\infty} M_0\left(1+\frac{r}{n}\right)^{nt} = M_0e^{rt}$$

1.1 Example

Consider f(x) = $\frac{(x^2 - 1)}{(x - 1)}$

Let's work it out for x = 1

We have: $\frac{(1^2 - 1)}{(1 - 1)}$ = 0/0

The answer cannot be defined.

Let's see the change in the function as we move towards 1.

1.1 Example

Consider the calculations the table below:

From the table we see the as the value of x approaches 1, the value of the function approaches 2.

This table can be thus, better represented with limits as

$$\lim_{x \to 1} \frac{(x^2 - 1)}{(x - 1)} = 2$$

x	
0.5	1.50000
0.9	1.90000
0.99	1.99000
0.999	1.99900
0.9999	1.99990
0.99999	1.99999
…	…

1.1 Example

The solving we saw is still incomplete.

As 1 can be approached from both sides, we need to consider the second way.

If the answer from both the sides is same, then we say that the function is continuous.

x		x	
0.5	1.50000	1.5	2.50000
0.9	1.90000	1.1	2.10000
0.99	1.99000	1.01	2.01000
0.999	1.99900	1.001	2.00100
0.9999	1.99990	1.0001	2.00010
0.99999	1.99999	1.00001	2.00001
…	…	…	…

2 Definition of Limit

Let f(x) be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function f(x) approach the real number L as the values of x($\neq$ a) approach the number a, then we say that the limit of f(x) as x approaches a is L. (More simply, as x gets closer to a, f(x) gets closer and stays close to L.)

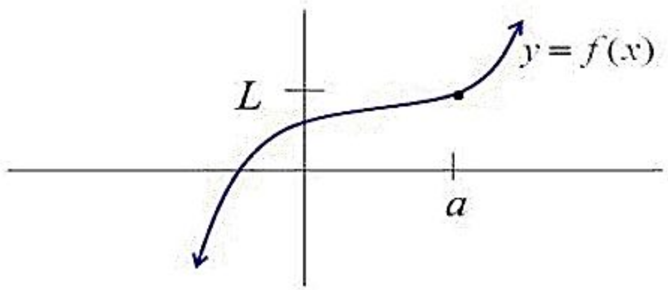
Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L$$

We say that the limit of $f(x)$ as x approaches a is L and write

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ approach L as x approaches a.



3 One - sided Limits

Limit from the left: Let f(x) be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function f(x) approach the real number L as the values of x (where x < a) approach the number a, then we say that L is the limit of f(x) as x approaches a from the left.
Symbolically, we express this idea as $\lim_{x \to a^-} f(x)$ = L

Limit from the right: Let f(x) be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function f(x) approach the real number L as the values of x (where x > a) approach the number a, then we say that L is the limit of f(x) as x approaches a from the right.
Symbolically, we express this $\lim_{x \to a^+} f(x)$ = L

3 One - sided Limits

One-sided limits are helpful when we want to check for discontinuities and if we want to confirm if a limit exists.

When does a limit not exist?

The $\lim_{x \to a} f(x)$ does exist if;

1. $\lim_{x \to a^-} f(x)$ exists. In other words there must be a limit from the left.

2. $\lim_{x \to a^+} f(x)$ exists. In other words there must be a limit from the right.

3. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$. The limit from the left must equal the limit from the right.

3 When it is different from different sides

How about a function **f(x)** with a "break" in it like this:

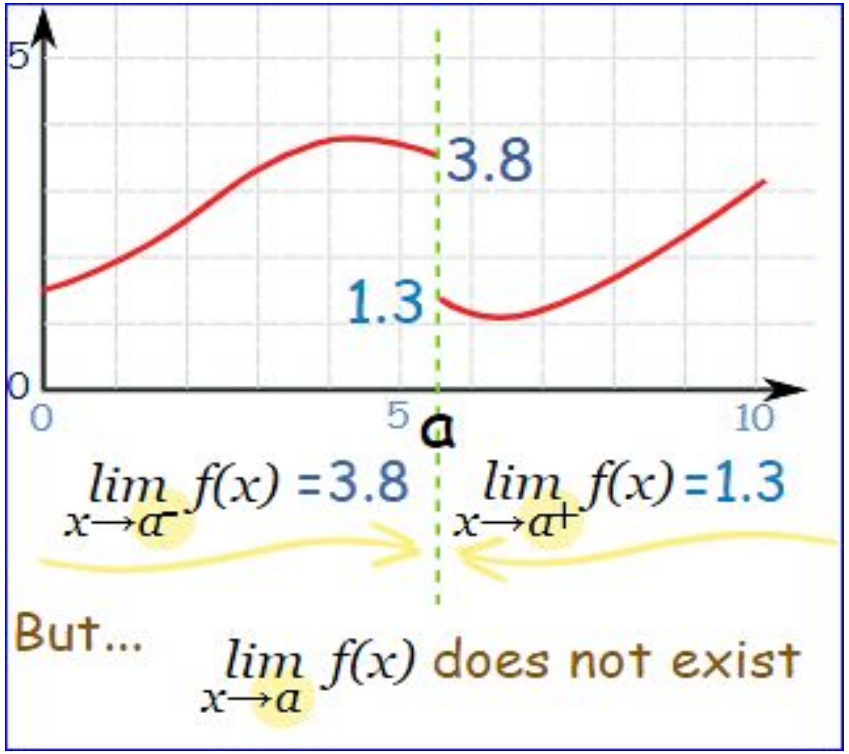
The limit does not exist at "a"

We can't say what the value at "a" is, because there are two competing answers:
3.8 from the left, and
1.3 from the right

But we **can** use the special "−" or "+" signs (as shown) to define one sided limits:

the **left-hand** limit (−) is 3.8
the **right-hand** limit (+) is 1.3

And the ordinary limit **"does not exist"**



3 Examples

Ex 1:

Consider the function $f(x) = \sqrt{x}$, $x \geq 0$.

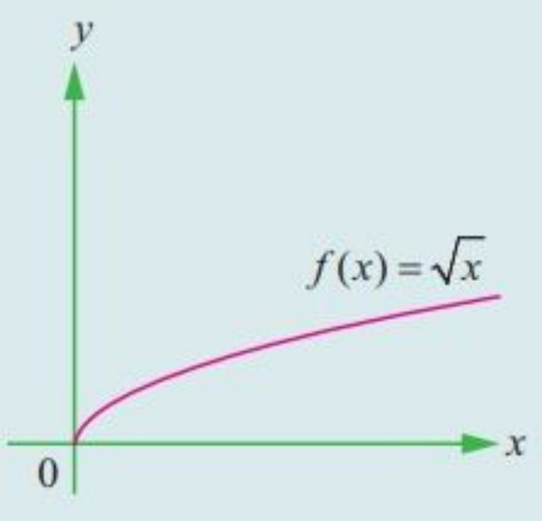
Does $\lim_{x \to 0} f(x)$ exist?

Solution

No. $f(x) = \sqrt{x}$ is not even defined for $x < 0$.

Therefore as $x \to 0^-$, $\lim_{x \to 0^-} \sqrt{x}$ does not exist.

However, $\lim_{x \to 0^+} \sqrt{x} = 0$. Therefore $\lim_{x \to 0} \sqrt{x}$ does not exist.



4 Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behaviour of a function around a given value. As we shall see, we can also describe the behaviour of functions that do not have finite limits.

We now turn our attention to h(x) = $\frac{1}{(x-2)^2}$

From its graph we see that as the values of x approach 2, the values of h(x) become larger and larger and, in fact, become infinite.

Mathematically, we say that the limit of h(x) as x approaches 2 is positive infinity.

Symbolically, we express this idea as $\lim_{x \to 2} h(x) = +\infty$.

4 Infinite Limits

Let f(x) be defined for all x ≠ a in an open interval containing a.

i) If the values of f(x) increase without bound as the values of x (where x ≠ a) approach the number a, then we say that the limit as x approaches a is positive infinity and we write $\lim_{x \to a} f(x) = +\infty$.

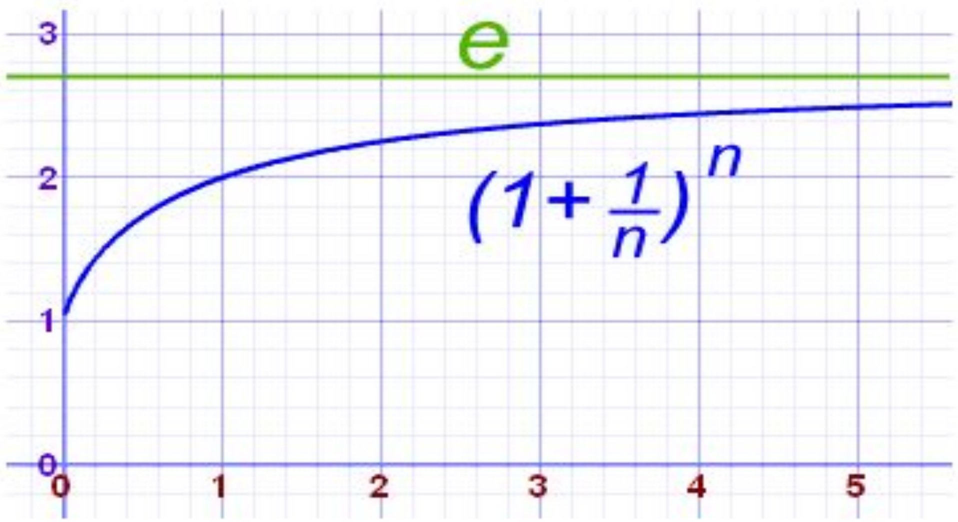
ii) If the values of f(x) decrease without bound as the values of x (where x ≠ a) approach the number a, then we say that the limit as x approaches a is negative infinity and we write $\lim_{x \to a} f(x) = -\infty$.

But don't be fooled by the "=". We cannot actually **get** to infinity, but in "limit" language the **limit is infinity** (which is really saying the function is limitless).

4 Example

Consider the function $(1+\frac{1}{n})^n$
What is the value of the function as n approaches infinity??

n	$(1 + 1/n)^n$
1	2.00000
2	2.25000
5	2.48832
10	2.59374
100	2.70481
1,000	2.71692
10,000	2.71815
100,000	2.71827



$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

5 Limits for Rational Functions

A Rational Function is one that is the ratio of two polynomials: f(x) $= \frac{P(x)}{Q(x)}$

To know limits to infinity of such rational functions what do we do?

Compare the Degree of P(x) to the Degree of Q(x).

1. If the Degree of P is **less than** the Degree of Q, the limit is 0.

2. If the Degree of P and Q are **the same**, divide the coefficients of the terms with the largest exponent.

3. If the Degree of P is **greater than** the Degree of Q, then the limit is positive infinity or maybe negative infinity. (We can work out the sign (positive or negative) by looking at the signs of the terms with the largest exponent)

Question

Evaluate $\lim_{x \to \infty} \frac{-4x^3 + 7}{2x^2 - 5x + 6}$

A. -∞
B. -2
C. 0
D. ∞

Solution

This is a rational function of the form $\dfrac{P(x)}{Q(x)}$ with deg(P) > deg(Q).

Then the limit is either positive infinity or negative infinity.

We need to look at the signs.

– $4x^3$ (the term with the largest exponent on the top) is negative
and $2x^2$ (the term with the largest exponent on the bottom) is positive.

negative ÷ positive is negative

So $\lim\limits_{x\to\infty} \dfrac{-4x^3+7}{2x^2-5x+6} = -\infty$

6 Theorems of limits

Theorem 1:
Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n$ be a polynomial, where $a_0, a_1, \ldots, a_n$ are real numbers and n is a fixed positive integer. Then:

$$\lim_{x\to x_0} P(x) = a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n = P(x_0)$$

Theorem 2:
The limit of a constant function is that constant.

Theorem 3:

If $\lim_{x\to x_0} f(x)$ exists then $\lim_{x\to x_0}[f(x)]^n$ exists and $\lim_{x\to x_0}[f(x)]^n = \left[\lim_{x\to x_0} f(x)\right]^n$.

Theorem 4:

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

6

Theorems of limits

Theorem 5:

(i) $\lim_{x \to x_0} cf(x) = c \lim_{x \to x_0} f(x)$,

(ii) $\lim_{x \to x_0} [f(x) \pm g(x)] = \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x)$,

(iii) $\lim_{x \to x_0} [f(x).g(x)] = \lim_{x \to x_0} f(x) . \lim_{x \to x_0} g(x)$ and

(iv) $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$, provided $\lim_{x \to x_0} g(x) \neq 0$.

7 Standard Limits formulae

$$\lim_{x \to 0} e^x = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$$

$$\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$$

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$$

Question

1. Calculate $\lim_{x\to 3}(x^3 - 2x + 6)$.

2. Calculate $\lim_{x\to x_0}(5)$ for any real number x_0.

3. Compute $\lim_{x\to 1}\frac{x^3 - 1}{x - 1}$.

Solution

Answer 1. $P(x)=x^3-2x+6$ is a polynomial.

Hence, $\lim_{x\to 3} P(x)=P(3)=3^3-2\times3+6=27.$

Answer 2. $\lim_{x\to x_0}(5)=f(x_0)=5$

Answer 3. $\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{x^3-1^3}{x-1}=3(1)^{3-1}=3$

8 What is continuity?

A function is said to be continuous in a given interval if there is no break in the graph of the function in the entire interval range.
If the left-hand limit, right-hand limit and the value of the function at x = c exist and are equal to each other, i.e.

$$\lim_{x \to c-} f(x) \; f(c) = \lim_{x \to c+} f(x)$$

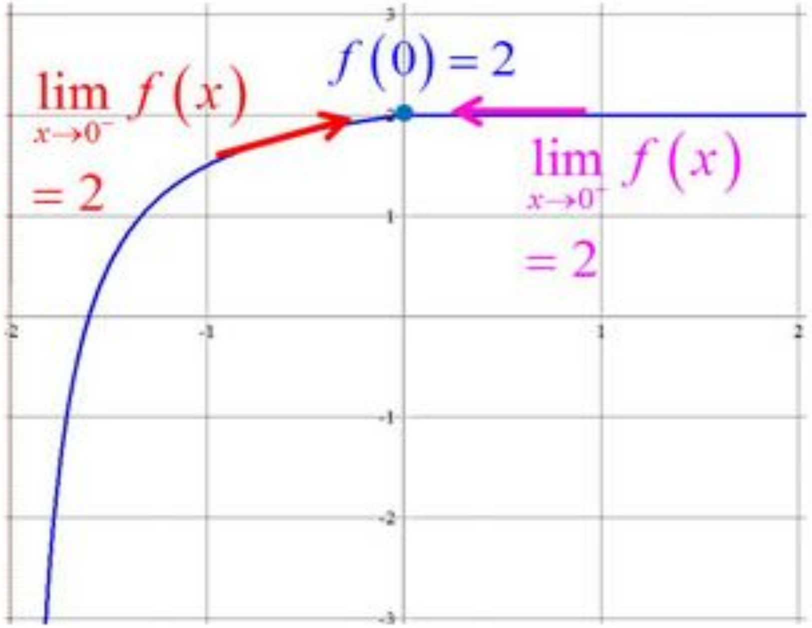
then f is said to be continuous at x = c

8.1 Conditions for continuity

1. The function is defined at $x = a$; that is, $f(a)$ equals a real number
2. The limit of the function as x approaches a exists
3. The limit of the function as x approaches a is equal to the function value at $x = a$

8.1 Example

Is f(x) continuous at x = 0? (Refer the graph)



Solution:

To check for continuity at $x = 0$, we check the three conditions:

1. Is the function defined at $x = 0$? Yes, $f(0) = 2$
2. Does the limit of the function as x approaches 0 exist? Yes
3. Does the limit of the function as x approaches 0 equal the function value at $x = 0$? Yes

Since all three conditions are met, $f(x)$ is continuous at $x = 0$.

8.1 Example

$$f(x) = \frac{x^2 - x - 2}{x + 1}$$ **is the function continuous at x = 2?**

Solutions

1. First check if the function is defined at x = 2.

$$f(2) = \frac{2^2 - 2 - 2}{2 + 2}$$

$$f(2) = \frac{0}{4}$$

$$f(2) = 0$$

8.1 Example

2.Checking the one-sided limits,

$$\lim_{x \to 2^+} \frac{x^2 - x - 2}{x + 1} = \frac{2^2 - 2 - 2}{2 + 2} = 0$$

$$\lim_{x \to 2^-} \frac{x^2 - x - 2}{x + 1} = \frac{2^2 - 2 - 2}{2 + 2} = 0$$

Since the one-sided limits agree, the limit exists. Since the limit is equal to the function value,

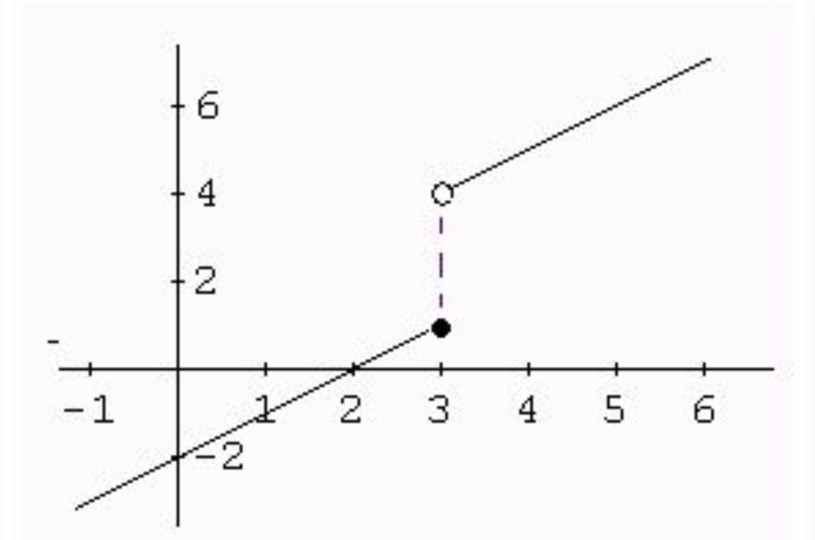
The function is continuous at $x = 2$.

9

What is discontinuity?

The function of the graph which is not connected with each other is known as a discontinuous function. A function f(x) is said to have a discontinuity of the first kind at x = a, if the left-hand limit of f(x) and right-hand limit of f(x) both exist but are not equal. f(x) is said to have a discontinuity of the first kind from the left at x = a, if the left hand of the function exists but not equal to f(a).

In the graph, the limits of the function to the left and to the right are not equal and hence the limit at x = 3 does not exist anymore. Such function is said to be a discontinuity of a function.



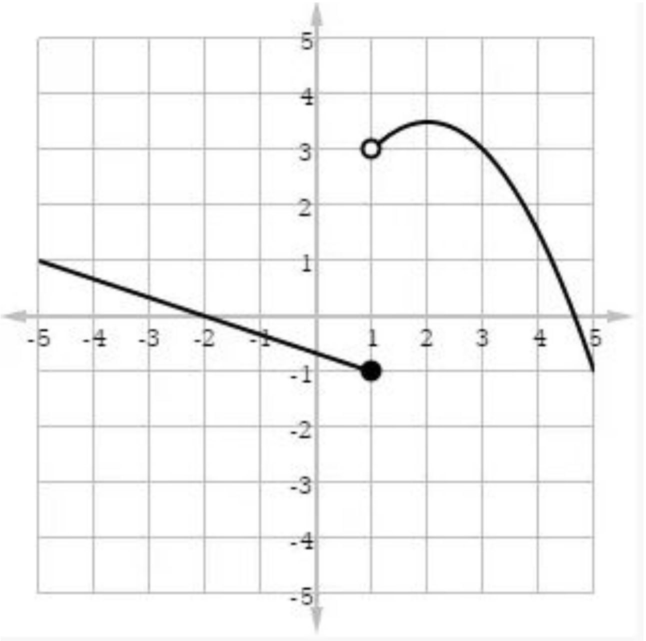
9.1 Types of discontinuity

1) Jump Discontinuities
One way in which a limit may fail to exist at a point *x* = *a* is if the left hand limit does not match the right hand limit.

Example :

$$\lim_{x \to 1^-} f(x) = -1$$

$$\lim_{x \to 1^+} f(x) = 3$$



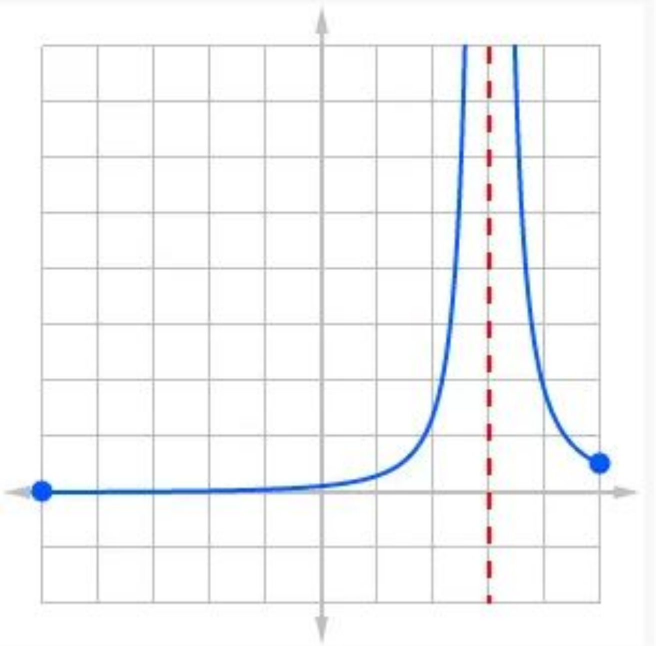
Because the left and right limits do not agree, the limit of f(x) as x → 1 does not exist.

Therefore, by definition, the function f is discontinuous at x = 1. This kind of discontinuity is known as a jump (for obvious reasons).

9.1 Types of discontinuity

2) Infinite Discontinuities (Vertical Asymptotes)

In some functions, the values of the function approach ∞ or -∞ as x approaches some finite number a. In this case, we say that the function has an infinite discontinuity or vertical asymptote at x = a.

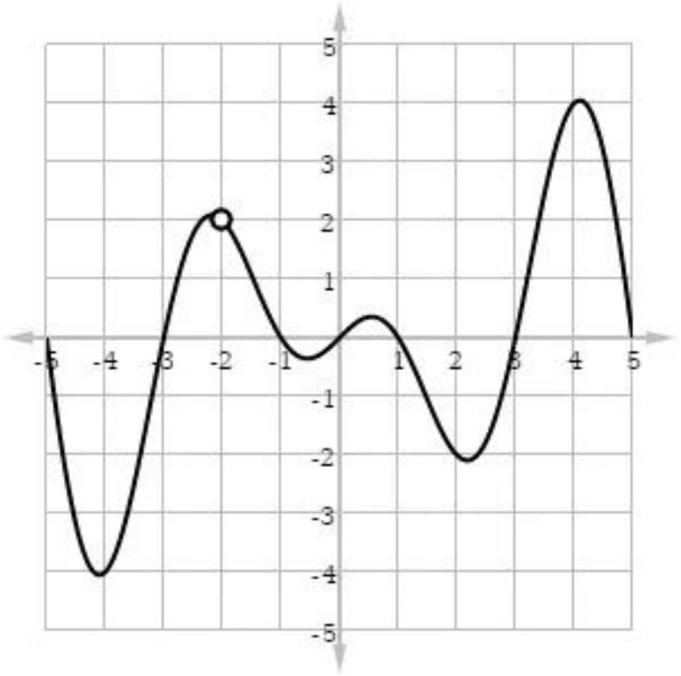


This function has an infinite discontinuity at $x = 3$.

9.1 Types of discontinuity

3) Removable Discontinuities

A function f has a removable discontinuity at $x = a$ if the limit of $f(x)$ as $x \to a$ exists, but either $f(a)$ does not exist, or the value of $f(a)$ is not equal to the limiting value.



Example :

There is a hole at $x = -2$. In fact, the graph would be continuous at that point if the hole at $(-2, 2)$ were filled in. That's the clue that we're dealing with a *removable* discontinuity at $x = -2$.

9.1 Example

Example 1: Identify if the function f(x) = (x – 2)/(x – 4) is a discontinuous function.

Solution:
As we can see, the function f(x) = (x – 2)/(x – 4) is not defined at x = 4.
Hence it is discontinuous at x = 4.
f(x) = (x – 2)/(x – 4) is a discontinuous function.

9.2 Which functions are continuous?

Some Typical Continuous Functions:

1. Trigonometric Functions in certain periodic intervals (sin x, cos x, tan x etc.)
2. Polynomial Functions (x^2 +x +1, x^4 + 2.... etc.)
3. Exponential Functions (e^{2x}, $5e^x$ etc.)
4. Logarithmic Functions in their domain ($\log_{10}x$, ln x^2 etc.)

10 Why is the study of continuity important?

- Calculus and analysis (more generally) study the behavior of functions, and continuity is an important property because of how it interacts with other properties of functions.

- In economics: Continuity simply means that there are no ‘jumps’ in people’s preferences. In mathematical terms, if we prefer point A along a preference curve to point B, points very close to A will also be preferred to B.

- Continuous functions on a compact set have the important properties of possessing maximum and minimum values and being approximated to any desired precision by properly chosen polynomial series, Fourier series, or various other classes of functions

11 Relationship between limits and continuity

- A limit is a certain value
- In limits, the value the function approaches from each side, and then we use this to find the actual limit.

- Continuity describes the *behavior* of a function.
- Continuity describes the behavior of a function at a certain point or section. We can use the limit to find the continuity.

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Question

1. Determine if the given function is continuous or discontinuous at the indicated points.

$$f(x) = \frac{4x+5}{9-3x}$$

(a) $x = -1$ (b) $x = 0$ (c) $x = 3$

Solution

1. (a) $$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{4x+5}{9-3x} = \frac{\lim\limits_{x\to -1}(4x+5)}{\lim\limits_{x\to -1}(9-3x)} = \frac{4\lim\limits_{x\to -1} x + \lim\limits_{x\to -1} 5}{\lim\limits_{x\to -1} 9 - 3\lim\limits_{x\to -1} x} = \frac{4(-1)+5}{9-3(-1)} = f(-1)$$

The function is continuous at x=–1.

1. (b) $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{4x+5}{9-3x} = \frac{\lim\limits_{x\to 0}(4x+5)}{\lim\limits_{x\to 0}(9-3x)} = \frac{4\lim\limits_{x\to 0} x + \lim\limits_{x\to 0} 5}{\lim\limits_{x\to 0} 9 - 3\lim\limits_{x\to 0} x} = \frac{4(0)+5}{9-3(0)} = f(0)$$

The function is continuous at x=0.

1. (c) $\lim_{x\to 3^-} f(x) = \infty$ $\lim_{x\to 3^+} f(x) = -\infty$

The function is not continuous at x=3.