Calculus

Assignment 1

Q1) Evaluate the following limit.

$$\lim_{h \to 0} \frac{2(-3+h)^2 - 18}{h}$$

Sol^n:- $$\lim_{h \to 0} \frac{2(9 - 6h + h^2) - 18}{h}$$

$$= \lim_{h \to 0} \frac{18 - 12h + 2h^2 - 18}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 - 12h}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}(h-6)}{\cancel{h}}$$

$\therefore$ by taking limit $h \to 0$

$$= 2(0-6)$$

$$= \underline{\underline{-12}}$$

Q2)

ⓐ $\lim_{x \to 4^+} g(x) = 2x$ $\qquad$ $\lim_{x \to 4^-} g(x) = 2x$

$= 2(4)$ $\qquad$ $= 2(4)$

$= \underline{8}$ — ① $\qquad$ $= 8$ — ②

equation ① is equal to equation ② the function is
<u>continuous at (a) $x = 4$</u>

ⓑ $\lim_{x \to 6^+} g(x) = x - 1$ $\qquad$ $\lim_{x \to 6^-} g(x) = 2x$

$= 6 - 1$ $\qquad$ $= \underline{2(6)}$

$= 5$ — ① $\qquad$ $= 12$ ②

equation ① is not equal to equation ② the function is
<u>discontinuous at (b) $x = 6$</u>

Q3) Solve the following limit

$$\lim_{x \to 9} \frac{x-9}{3-\sqrt{x}}$$

Sol^n:- $\lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{(3-\sqrt{x})(\sqrt{x}+3)}$

$= \lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{-(x-9)}$

$= \lim_{x \to 9} -\sqrt{x} - 3$

by taking limit $x \to 9$

$= -\sqrt{9} - 3$

$= \underline{-6}$

Q(4) Determine if the following function is continuosus or discontinuos at

ⓐ $x = -1$

Sol^n :-

$$\lim_{x \to (-1)^-} \frac{4x+5}{9-3x} = \frac{4(-1)+5}{-3(-1)+9} = \frac{1}{12} \quad — ①$$

$$\lim_{x \to (-1)^+} \frac{4x+5}{9-3x} = \frac{4(-1)+5}{-3(-1)+9} = \frac{1}{12} \quad — ②$$

∵ opposite direction limits are same function is continuous.

ⓑ $x = 0$

Sol^n :-

$$\lim_{x \to (0)^-} \frac{4x+5}{9-3x} = \frac{4(0)+5}{9-3(0)} = \frac{5}{9} \quad — ①$$

$$\lim_{x \to (0)^+} \frac{4x+5}{9-3x} = \frac{4(0)+5}{9-3(0)} = \frac{5}{9} \quad — ②$$

∵ Opposite direction limits are same function is continuous.

ⓒ $x = 3$

Sol^n :-

$$\lim_{x \to (3)^-} \frac{4x+5}{9-3x} = \lim_{x \to (3)^-} (4x+5) \lim_{x \to (3)^-} \frac{1}{9-3x} = 17\,\infty = \infty$$

$$\lim_{x \to (3)^+} \frac{4x+5}{9-3x} = \lim_{x \to (3)^+} (4x+5) \lim_{x \to (3)^+} \frac{1}{9-3x} = 17(-\infty) = -\infty$$

∵ Opposite direction limits are not same function is discontinuous.

Q5) Evaluate each of the following limits

$$\lim_{x\to\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}}$$

Sol^n :- $\lim_{x\to\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}}$

$= \frac{\cancel{6 - 1/e^{6x}}}{\cancel{8 - e}}$

$\lim_{x\to\infty} \frac{6 - e^{1/6x}}{8 - e^{1/2x} + e^{3/5x}}$ ... (Divide by highest denominator power)

$= \frac{\lim_{x\to\infty}(6 - e^{1/6x})}{\lim_{x\to\infty}(8 - e^{1/2x} + e^{3/5x})}$

$= \frac{6}{8}$ ... (taking limit)

$= \frac{3}{4}$

Q6) Given function

$$g(y) = \begin{cases} y^2+5 & \text{if } y < -2 \\ 1-3y & \text{if } y \geq -2 \end{cases}$$

ⓐ $\lim_{y \to 6} g(y) = 1-3y$

$= 1-(3)(6)$

$= \underline{-17}$

(b) $\lim_{y \to -2^+} g(y) = 1 - 3y$

$= 1 - 3(-2)$

$= \underline{\underline{7}}$

$\lim_{y \to -2^-} g(y) = y^2 + 5$

$= (-2)^2 + 5$

$= \underline{\underline{9}}$

Q7) Use the product Rule to find the derivative of.

$f(t) = (4t^2 - t)(t^3 - 8t^2 + 12)$

Solⁿ:- By applying product rule $(f \cdot g)' = f' \cdot g + f \cdot g'$

here $f = 4t^2 - t \quad g = t^3 - 8t^2 + 12$

$\therefore$

$$= \frac{d}{dx}(4t^2 - t)(t^3 - 8t^2 + 12) + \frac{d}{dx}(t^3 - 8t^2 + 12)(4t^2 - t)$$

$$= (8t - 1)(t^3 - 8t^2 + 12) + (3t^2 - 16t)(4t^2 - t)$$

$$= 20t^4 - 132t^3 + 24t^2 + 96t - 12$$

$$= \underline{\underline{2(10t^4 - 66t^3 + 12t^2 + 48t - 6)}}$$

Q8) Use the Quotient Rule to find the derivative of;

$$R(\omega) = \frac{3\omega + \omega^4}{2\omega^2 + 1}$$

Solⁿ:- By applying Quotient Rule $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}$

Here, $f = (3\omega + \omega^4)$ $\quad g = (2\omega^2 + 1)$

$$\therefore \quad = \frac{d/dx\,(3\omega + \omega^4)(2\omega^2 + 1) - d/dx\,(2\omega^2 + 1)(3\omega + \omega^4)}{(2\omega^2 + 1)^2}$$

$$= \frac{(3 + 4\omega^3)(2\omega^2 + 1) - 4\omega(3\omega + \omega^4)}{(2\omega^2 + 1)^2}$$

$$= \frac{4\omega^5 + 4\omega^3 - 6\omega^2 + 3}{(2\omega^2 + 1)^2}$$

Q9) Differentiate :-

$f(x) = 2e^x - 8^x$

Sol^n :- $= \dfrac{d}{dx}(2e^x) - \dfrac{d}{dx}(8^x)$

$= 2\dfrac{d}{dx} e^x - 8^x \ln(8)$

$= \underline{\underline{2e^x - 3\ln(2)\, 8^x}}$

Q10) Differentiate :-

$$y = z^5 - e^z \ln(z)$$

Sol^n :-

$$= \frac{d}{dx}(z^5) - \frac{d}{dx}(e^z \ln(z))$$

$$= 5z^4 - \frac{d}{dx}(e^z)\ln(z) + \frac{d}{dx}(\ln(z))e^z$$

$$= 5z^4 - (e^z)\ln(z) + \frac{e^z}{z}$$

Q 11) Using chain rule Differentiate the following :-

ⓐ $f(x) = (6x^2 + 7x)^4$

Solⁿ :- $= \frac{d}{dx}(6x^2+7x)^4$

$= 4(6x^2+7x)^3 \frac{d}{dx}(6x^2+7x)$ . . . (by applying chain rule)

$= \underline{\underline{4(6x^2+7x)^3 (12x+7)}}$

ⓑ $f(t) = 5 + e^{(4t+t^7)}$

Solⁿ :- $= \frac{d}{dx}\left(5 + e^{(4t+t^7)}\right)$

$= \frac{d}{dx}(5) + \frac{d}{dx}\left(e^{4t+t^7}\right)$

$= 0 + \left(e^{4t+t^7}\right)\frac{d}{dx}(4t+t^7)$

$= 0 + \left(e^{4t+t^7}\right)(4+7t^6)$

$=$

$= e = \underline{\underline{e^{t^7+4t}(7t^6+4)}}$

Q12) Determine the second derivative of

(a) $z = \ln(7-x^3)$

Sol^n:- $= \dfrac{d^2}{dx^2}\left(\ln(7-x^3)\right)$

$$= \frac{d}{dx}\left(-\frac{3x^2}{7-x^3}\right)$$

$$= -3\frac{d}{dx}\left(\frac{x^2}{7-x^3}\right)$$

$$= \frac{-3\; d/dx\,(x^2)(7-x^3) - d/dx\,(7-x^3)\,x^2}{(7-x^3)^2}$$

.... (by using quotient rule)

$$= \frac{-3 \cdot 2x(7-x^3) - (-3x^2)x^2}{(7-x^3)^2}$$

$$= -\frac{3(14x + x^4)}{(7-x^3)^2}$$

(b) $Q(v) = \dfrac{2}{(6+2v-v^2)^4}$

Solⁿ:- $= \dfrac{d^2}{dx^2}\left[\dfrac{2}{(6+2v-v^2)^4}\right]$

$= \dfrac{d}{dx}\left[\dfrac{-8(-2v+2)}{(6+2v-v^2)^5}\right]$

$= -8\dfrac{d}{dx}\left[\dfrac{-2v+2}{(16+2v-x^2)^5}\right]$

$= -8 \dfrac{d/dx(-2v+2)(16+2v-v^2)^5 - d/dx\left[(16+2v-v^2)^5\right](-2v+2)}{\left[(16+2v-v^2)^5\right]^2}$

$= -8\dfrac{(-2)(16+2v-v^2)^5 - 5(16+2v-v^2)^4(-2v+2)(-2v+2)}{\left[(16+2v-v^2)^5\right]^2}$

$= \dfrac{16(9v^2-18v+26)}{(-v^2+2v+16)^6}$

(c) $f(t) = \ln(1+t^2)$

Sol^n :- $= \dfrac{d^2}{dx^2}\left[\ln(1+t^2)\right]$

$$= \frac{d}{dx}\left[\frac{2\cancel{x}t}{1+\cancel{x}t^2}\right]$$

$$= \frac{2(1-t^2)}{(1+t^2)^2}$$

$$= \frac{2-2t^2}{(1+t^2)^2}$$

Q13) Find the maximum and minimum value of the function $\left[x^3 - 3x^2 - 9x + 12\right]$

Sol^n:- Here,

$$y = \left[x^3 - 3x^2 - 9x + 12\right]$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(x^3 - 3x^2 - 9x + 12\right)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(3x^2) - \frac{d}{dx}(9x) + \frac{d}{dx}(12)$$

$$= 3x^2 - 6x - 9 + 0$$

$$= 3x^2 - 6x - 9$$

$\because$ For stationary values, $\frac{dy}{dx} = 0$

$\therefore$

$$3x^2 - 6x - 9 = 0$$

$$3(x^2 - 2x - 3) = 0$$

$$3(x^2 + x - 3x - 3) = 0$$

$$3(x(x+1) - 3(x+1)) = 0$$

$$3(x+1)(x-3) = 0$$

$\therefore$ $(x+1) = 0$ or $(x-3) = 0$

$x = -1$ or $x = 3$

$\therefore$ At $x = -1$ and $x = 3$ we get stationary values.

Now, $\frac{d^2y}{dx^2} = 3x^2 - 6x - 9$

$\frac{d}{dx}(3x^2 - 6x - 9)$

$= \frac{d}{dx}(3x^2) - \frac{d}{dx}(6x) - \frac{d}{dx}(9)$

$= 6x - 6 - 0$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=-1} = 6[(-1)] - [6]$

$= -6 - 6$

$= -12$ (negative)

$\therefore$ At $x = -1$ the function is maximum.

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=3} = 6 \times 3 - 6$

$= 18 - 6$

$= 12$ (positive)

$\therefore$ At $x = 3$ the function is minimum

Now, $y = x^3 - 3x^2 - 9x + 12$

putting $x = -1$

$y_{max} = (-1)^3 - 3(-1)^2 - 9(1) + 12$

$= -1 - 3 + 9 + 12$

$y_{max} = \underline{\underline{17}}$

putting $x = 3$

$y_{min} = 3^3 - 3 \cdot (3)^2 - 9 \times 3 + 12$

$= 27 - 27 - 27 + 12$

$y_{min} = \underline{\underline{-15}}$

Q14) Find the maximum and minimum value of the function $[4x^3 - 18x^2 + 24x - 7]$

Sol^n: Here,

$$y = 4x^3 - 18x^2 + 24x - 7$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(4x^3 - 18x^2 + 24x - 7)$$

$$= \frac{d}{dx}(4x^3) - \frac{d}{dx}(18x^2) + \frac{d}{dx}(24x) - \frac{d}{dx}(7)$$

$$= 12x^2 - 36x + 24 - 0$$

$$= 12x^2 - 36x + 24$$

For stationary values, $\frac{dy}{dx} = 0$

$$\Rightarrow 12x^2 - 36x + 24 = 0$$

$$12(x^2 - 3x + 2) = 0$$

$$12(x^2 - x - 2x + 2) = 0$$

$$12(x(x-1) - 2(x-1)) = 0$$

$$12(x-1)(x-2) = 0$$

$$(x-1) = 0 \quad \text{or} \quad (x-2) = 0$$

$$\Rightarrow x = 1 \quad \text{or} \quad x = 2$$

$\therefore$ At $x = 1$ and $x = 2$ we get stationary values

Now $\frac{d^2y}{dx^2} = 12x^2 - 36x + 24$

$$\frac{d}{dx}(12x^2 - 36x + 24)$$

$$= \frac{d}{dx}(12x^2) - \frac{d}{dx}(36x) + \frac{d}{dx}(24)$$

$$= 24x - 36$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1} = 24(1) - 36$$

$$= 24 - 36$$

$$= -12 \text{ (negative)}$$

$\therefore$ At $x = 1$ the function is maximum

$$\left(\frac{d^2y}{dx^2}\right)_{x=2} = 24(2) - 36$$

$$= 48 - 36$$

$$= 12 \text{ (positive)}$$

$\therefore$ At $x = 2$ the function is minimum

Now, $y = 4x^3 - 18x^2 + 24x - 7$

putting $x = 1$

$$y_{max} = 4\cdot(1)^3 - 18(1)^2 + 24(1) - 7$$

$$= 4 - 18 + 24 - 7$$

$$y_{max} = \underline{\underline{3}}$$

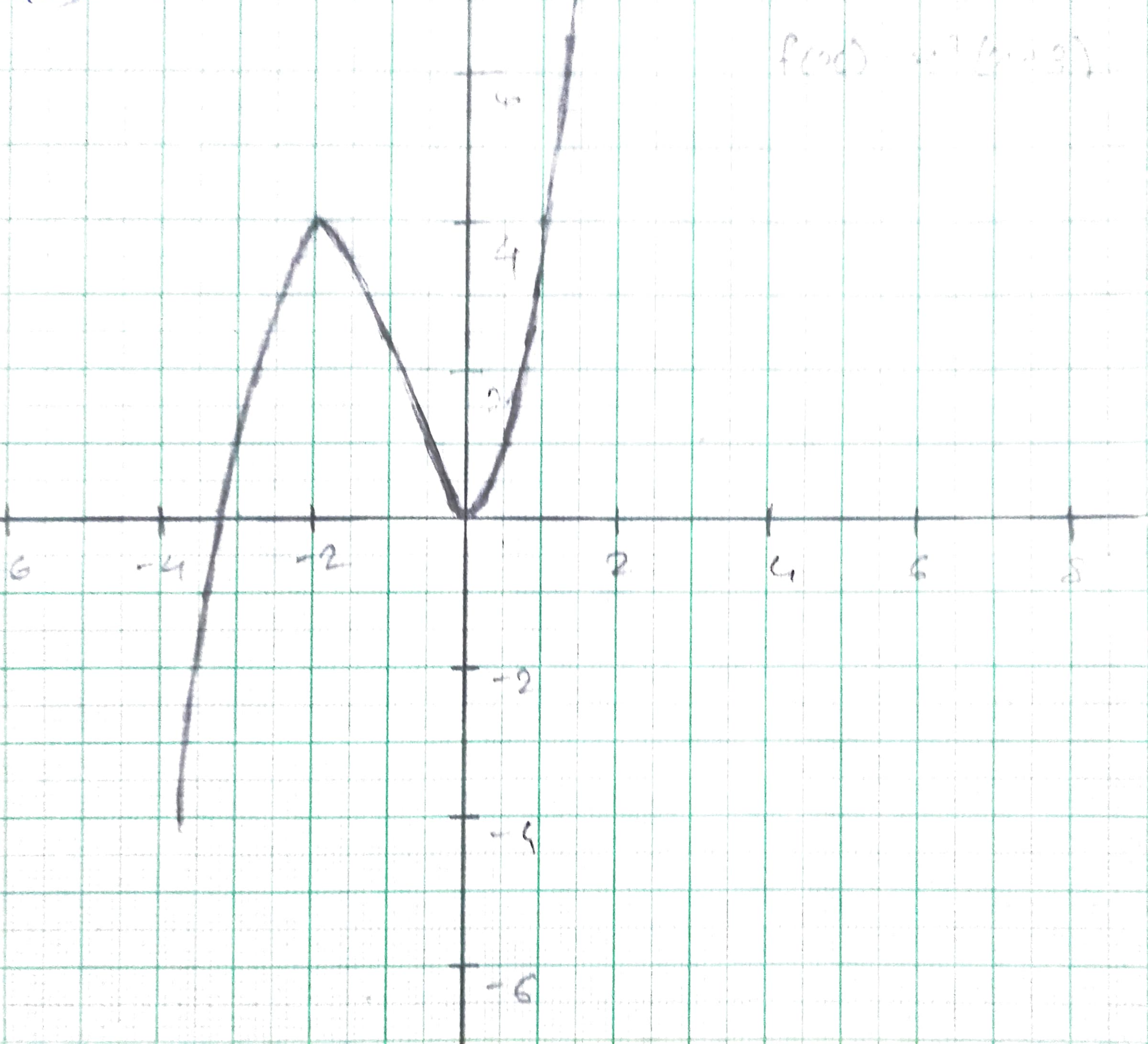
putting $x = 2$

$$y_{min} = 4\cdot(2)^3 - 18(2)^2 + 24(2) - 7$$

$$= 32 - 72 + 48 - 7$$

$$y_{min} = \underline{\underline{1}}$$

Q15)



Q16) Sketch a detailed graph for the following function

$f(x) = 3x^2 - 6x$

Solⁿ:- $f(x) = 3x^2 - 6x$

compairing with $ax^2 + bx + c$

$a = 3$, $b = -6$, $c = 0$

1) Consider the vertex form of a parabolla

$a(x+d)^2 + e$ . . . . — ①

here, $d = \frac{b}{2a}$

$\therefore d = \frac{-6}{2 \times 3} = -1$ . . . (by substituting values) — ②

also $e = c - \frac{b^2}{4a}$

$= 0 - \frac{(-6)^2}{4 \times 3}$ . . . (by substituting values)

$= -\frac{36}{12} = -3$ ⟵ ③

by substituting values from ② & ③ in ①

let, $y = 3(x-1)^2 - 3$

since, $y = a(x-h)^2 + h$

by compairing $a = 3$, $h = 1$, $k = (-3)$

$\therefore$ vertex (h, k)

$= \underline{\underline{(1, -3)}}$

2) Now,

let p, be the distance from the vertex to the focus.

$\therefore \; p = \dfrac{1}{4a}$

$p = \dfrac{1}{4 \times 3} = \underline{\underline{\dfrac{1}{12}}}$

3) focus of the parabola

$f = (h, k + p)$

$f = \left(1, -3 + \dfrac{1}{12}\right)$

$= \underline{\underline{\left(1, \dfrac{-35}{12}\right)}}$

4) Axis of symmetry $= \underline{\underline{x = 1}}$

5) Directix of a parabola

$= y = k - p$

$y = -3 - \dfrac{1}{12} = \underline{\underline{-\dfrac{37}{12}}}$

Vertex	$(1, -3)$
Focus	$(1, -35/12)$
Axis of symmetry	$x = 1$
Directix	$y = -37/12$

Selecting few values of x to find the corresponding y values in given equation.

$\therefore$ if $x = 0$

$f(0) = 3(0)^2 - 6 \times 0$

$f(0) = 0$

$\therefore$ the y value at $x = 0$ is 0

$y = 0$

if $x = -1$

$f(-1) = 3(-1)^2 - 6(-1)$

$f(-1) = 9$

$\therefore$ the y value at $x = (-1)$ is 9

$y = 9$

$\therefore$ if $x = 2$

$f(2) = 3(2)^2 - 6 \times 2$

$= 12 - 12$

$= 0$

$\therefore$ the y value at $x = 2$ is 0

$y = 0$

if $x = 3$

$f(3) = 3(3)^2 - 6(3)$

$= 27 - 18$

$= 9$

$\therefore$ the y value at $x = (3)$ is 9

$y = 9$

$\therefore$ The selected points become

x	y
-1	9
0	0
1	-3
2	0
3	9

(Q16)