PRIYANSHU Roll No :- 432

FE1 : Assignment 2 Unit 3 & 4

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Q1. Given:-

$S_0 = 65$

$\sigma = 25\% / 0.25$ p.a

$r = 2\%$ p.a

$X = 55$

$T = 1/2$ or 6 months

Also, BSM applies

(i) $C_0 = ?$

$$C_0 = S_0 \Phi(d_1) - Xe^{-rT} \Phi(d_2) \quad -(1)$$

$\therefore$

$$d_1 = \frac{\ln \frac{S_0}{X} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln\left(\frac{65}{55}\right) + (0.02 + \frac{1}{2} \times 0.25^2)\frac{1}{2}}{0.25\sqrt{0.5}}$$

$d_1 = 1.08996$

$d_2 = d_1 - \sigma\sqrt{T}$

$d_2 = 1.08996 - 0.25\sqrt{0.5}$

$= 0.9132$

Putting the found values in Eq (1)

$$C_0 = 65 . \Phi(1.08996) - 55e^{-0.02(0.5)} . \Phi(0.9132)$$

$$= 65.(0.86214) - 55 e^{-0.02(0.5)} . \Phi(0.81859)$$

$C_0 = £11.4646$

(ii) Delta of Call option:-

Adelta :

$$\frac{\text{Change in Price of Call option Price}}{\text{Change in Price of Underlying Share}}$$

$$= \frac{dC}{dS_t}$$

(iii) Value of delta of Call option:-

$$\frac{dC}{dS} = \underline{\underline{0.86214}}$$

(iv) Value of ~~Rut~~ delta of Put option

$$\frac{dP}{dS} = -\Phi(-0.86214)$$

$$= -(1-0.86214)$$

$$= -0.13786$$

Q3

(i) Delta :-

↳ Delta of Option Δ, is defined as the rate of change of option price respected to rate of change of underlying asset price :

$$\Delta = \frac{d\Pi \rightarrow \text{option price}}{dS \rightarrow \text{underlying asset Price}}$$

Vega:-

ν is defined as rate of change of option price resp. to volatility of underlying asset

$$\nu = \frac{d\Pi}{d\sigma}$$

$\Pi \rightarrow$ option price [call/put]

$\sigma \rightarrow$ volatility of stock price

(iii) Provided:-

$S_0 = 55$, $X = 50$, $\sigma = 25\%$, $r = 5\%$ p.a, $T = 1$ yr.

[Continued on next Page]

(ii) From Put-Call Parity, we know that:

$$C - P = S - Ke^{-rT}$$

[differentiating the above wrt "Volatility"]

$$LHS = \frac{d}{d\sigma}(C-P) = \nu_c - \nu_p$$

$$RHS = \frac{d}{d\sigma}(S - Ke^{-rT})$$

[Since the RHS components are independent, thus RHS = 0]

Therefore, we conclude that given the same underlying features/characteristics, Vega for European put/call is same:-

$$\nu_c - \nu_p = 0 \Rightarrow \nu_c = \nu_p$$

$\rightarrow$

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Continued Part (iii)

we know,

$$C_0 = S_0\,\Phi(d_1) - X e^{-rT}\,\Phi(d_2)$$

$$\therefore\ d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln\left(\frac{55}{50}\right) + \left(0.05 + 0.5(0.25)^2\right)}{0.25\sqrt{1}}$$

$$d_1 = \underline{\underline{0.7062}}$$

$$d_2 = 0.7062 - \sigma\sqrt{T}$$
$$= \underline{\underline{0.4562}}$$

$$C_0 = 55\,\Phi(0.7062) - 50e^{-0.05}\,\Phi(d_2)$$
$$= 55 \times (0.76115) - 50e^{-0.05} \times (0.67724)$$

$$C_0 = \underline{\underline{\$9.6527}}$$

By Put - Call Parity :-

$$c - P = S_0 - Ke^{-rT}$$
$$9.6527 - P = 55 - 50e^{-0.05}$$

$$\boxed{P = 2.2142}$$

(iv) **Delta Hedging of Portfolio:-**

↳ It is an option strategy that seeks to be directly neutral by establishing option - share relⁿ.

Vega Hedging:-

↳ Such a portfolio is immune to small changes in the price of the underlying asset portfolio for which the overall vega.

(v) Use:-

European Call $= \$9.6527$

European Put & $= \$2.2142$

Forward Stock 1yr. [No value is given but we know that Delta is 1 & Vega is 0]

Thus, for Vega-Neutrality, vega of ~~Portfolio~~ forward must be 0.

The eq we get is: $x \cdot V_c + y \cdot V_p = 0$

Also, $V_c = V_p$

$\therefore (x+y) * V_c = 0$

$\therefore x+y = 0$

$\therefore y = -x$

Delta - Neutrality : Delta of a forward we know is 1.

Thus for delta neutrality we require:-

$$x(\Delta c) + y(\Delta p) + z = 0$$

$\because \Delta p = \Delta c - 1$ & $y = -x$

$\therefore x + z = 0$

$$\boxed{z = -x}$$

→

When we combine the above equations, the overall porfolio we get :-

$x = -y = -z$ } Entirely equal to \$1000

$\therefore x(c) + y(p) + z(0) = 1000$

$\therefore x(9.6527) \overset{-x}{+y} (2.2142) = 1000$

$\therefore x = \dfrac{1000}{7.4385} = \underline{\underline{134.44}}$

$\therefore y = z = \underline{\underline{-134.44}}$

Q.3

(ii) Given :-

$S_0 = 50$, $r = 5\%$ pa
$X = 49$, $\sigma = 25\%$ p.a
$T = 6$ months ~~as~~ $[0.5]$

We know that,

$C_0 = S_0 \cdot \Phi(d_1) - Xe^{-rT} \Phi(d_2)$

$d_1 = \dfrac{\ln\left(\frac{50}{49}\right) + \left(0.05 + \frac{1}{2}(0.25)^2\right) 0.5}{0.25\sqrt{0.5}}$

$= \underline{\underline{0.3441}}$

$d_2 = d_1 - \sigma\sqrt{T}$

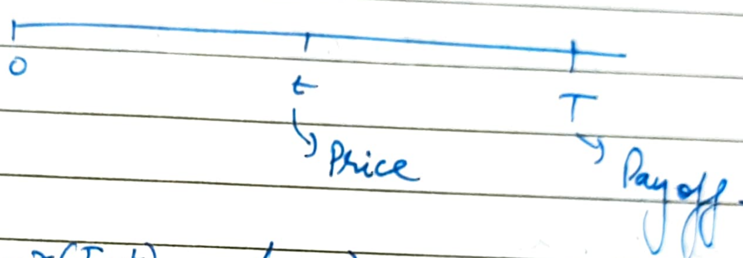
$= 0.3441 - 0.25\sqrt{0.5}$

$= \underline{0.1673}$

$C_0 = 50 \cdot \overbrace{\Phi(0.3441)}^{0.6346} - 49 \cdot e^{-0.05\times0.5} \cdot \overbrace{\Phi(0.1673)}^{0.5664}$

$= \$4.66$

(ii) We require C_t



$\therefore C_t = E\left(e^{-r(T-t)} C_T / F_t\right)$

where F_t denotes the filtration process

C_T is the payoff at time T.

C_t: derivative at time t with expectation taken under risk-neutral martingale measure.

(iii) Same as European Call since price for American & European Call is similar in value.

$\therefore C = \$4.66$,

(iv) European Put = ?

Using Put-Call parity,

$C - P = S - Ke^{-rT}$

$\rightarrow$

European Put = $2.4518

(v) Reasons ~~for~~ along with the change are:-

If the Stock is dividend-paying, the payment of the dividends would cause asset value to fall. [due to no-arbitrage principle].
This perhaps causes the European Call price to fall since its assumed that at Time 0 ~~the~~ holder doesn't own the asset.
In Puts, its the vice versa of the above causing the Put price to increase.

Q.4

(i) We know that,
$Z_t \sim SBM$ under real world measure P.
Also Y_t is a previsible process
$\therefore$ There exists a measure Q ~~and~~ equivalent to P
where $\tilde{Z}_t = Z_t + \int_0^t Y_s\, ds$ is SBM under Q.

Also, if Z_t is SBM under P and if Q is equivalent to P then there exists a previsible process Y_t such that $\tilde{Z}_t = Z_t + \int_0^t Y_s\, ds$ is BM under Q.

(ii) Under risk-neutral probab measure, the discounted value of asset prices are martingales.

(iii)

Q.5 (i) Delta = $N(d_1)$ i.e $E[S_t / S_t > S_0]$

$\therefore$ Δ delta : $\frac{\text{Change in Price of Call option Price}}{\text{Change in Price of Share.}}$

(ii) Given:-

$S_0 = 40$, $r = 2\%$ p.a

$X = 45.91$, $T = 5$, $d_1 = 0.6179$

$N(0.6179)$

$$\therefore d_1 = \frac{\ln\left(\frac{40}{45.91}\right) + \left(0.02 + 0.5(\sigma)^2\right)5}{\sigma\sqrt{5}}$$

$$\therefore N(0.6179) = \frac{-0.1378 + 0.1 + 2.5(\sigma)^2}{\sigma \times 2.2361}$$

$$0.3 \times 2.2361 = \frac{-0.0378 + 2.5\sigma^2}{\sigma}$$

$$0.67083\sigma = -0.0378 + 2.5\sigma^2$$

$$\underbrace{2.5}_{a}\sigma^2 \underbrace{-0.67083}_{b}\sigma \underbrace{-0.0378}_{c} = 0$$

$\therefore$ By Quadratic Eq formula:-

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-(-0.67083) \pm \sqrt{(-0.67083)^2 - 4(2.5 \times -0.0378)}}{2(2.5)}$$

$$= -0.0478 \text{ or } 0.316156$$

$\therefore$ $\sigma = 31.6156\% \sim \boxed{32\%}$

(iii) Given:-

$R_0 = 30$, R_t

$\sigma = \sqrt{0.15}$ p.a

c at time T if : $\frac{S_1}{S_0} < K_S$

$\frac{R_1}{R_0} < K_R$

Under risk-neutral probab q, fair price of option is:-

$$c \cdot e^{-rT} q\left(S_1/S_0 < K_S\right)\left(R_1/R_0 < K_R\right)$$

(iv) Under BSM, we know that:-
Perfectly Correlated case is ⇒

$$S_1/S_0 = R_1/R_0$$

① ∴ if $K_S < K_R$ then, option depends on S and so value:-

$$ce^{-rT} q\left(S_1/S_0 < K_S\right)$$

② Also, if $K_S > K_R$, then option depends on R and so:-

$$ce^{-rT} q\left(R_1/R_0 < K_R\right)$$

③ Also finally, if $K_S = K_R$ then option can be defined in terms of the price of either stock as

$$ce^{-rT} q\left(S_1/S_0 < K_S\right) = ce^{-rT} q\left(R_1/R_0 < \cancel{K_R} K_S\right)$$

(v) $ce^{-rT} \varphi(S_T/S_0 < K_S)\,\varphi(R_T/R_0 < K_R)$

$= 50e^{-0.02} \varphi(S_T/S_0 < 0.8)\,\varphi(R_T/R_0 < 0.6)$

$= 50e^{-0.02} \varphi(S_1 < 0.8 \times 40)\,\varphi(R_1 < 0.6 \times 30)$

$= 50e^{-0.02}(1 - \Phi((\log 1/0.8) + 0.02 - 0.5 \times 0.32^2/0.32) \times$
$(1 - \Phi((\log 1/0.6) + 0.02 - 0.5 \times 0.15/\sqrt{0.15})$

$= 50e^{-0.02}(1 - 0.59982)(1 - \Phi(1.1769))$

$= \boxed{\$1.61}$ [with $\sigma = 32\%$, found earlier]

Q.6 Given:-
$S_0 = 640$

(i) (a) $X = 630$
Short = 24830 Shares
$r = 3\%$ pa
BSM holds true

Since Δ for put $= -\Phi(-d_1)$

$\therefore \boxed{\Phi(d2) - 1}$

(b) In this case, we must have $100{,}000\,\Delta = -24{,}830$ and so $\Delta = -0.25$

(ii) As we know from earlier, $\Delta = -0.2483$ and so $\Phi(d_1) = \Phi(-0.25)$

$\therefore 1 - \Phi(d_1) = 0.25$
$1 - 0.25 = \Phi(d_1)$
$0.75 = \Phi(d_1)$
probab part.
$\therefore \boxed{d_1 = 0.68}$

Thus, by solving the d1 quadratic eq., we get :-

$\sigma = 0.68 \pm \sqrt{0.3709} = 0.07098$
$= \boxed{7.1\%}$

(iii) We need to Calculate $Ke^{-rT}\Phi(-d2) = Ke^{-r}\Phi(-d_1 + \sigma\sqrt{T})$
Thus, putting in the given values,

$630e^{-0.03}\Phi(-0.609)_p = 630e^{-0.03} \times 0.2712 = 165.806p$

Option Price =

$Xe^{-rT}\Phi(-d_2) - S_0\Phi(-d_1) \rightarrow$

~~$= 630e^{-0.03}\Phi(-0.609)_p = 630e^{-0.03} \times 0.2712 = 165.806p$~~

$\therefore\ 165.806 - \frac{24830 \times 640}{100000} = 6.894p.$

£ value of Cash holding [i.e amount I need to invest to hedge my option position X 100,000 units]

$\therefore\ 100000 \times 165.806p = £165806$

Q.7

(i) Let "h" be the individual derivative in following examples and "q" be the underlying asset.

$\therefore$ ① Delta:-

$$\Delta = \frac{dh}{dq} = \frac{dh}{dq}(t, S_t)$$

→ time

→ Stock at time t

② $\Gamma = \frac{d^2h}{dq^2}$ [∵ Gamma is double deriv. of delta due to delta-hedging feature]

③ $\nu = \frac{dh}{d\sigma}$ → volatility

→ (ii) Given:-

$X = 50,\ c = 17.91,\ S_0 = 60,\ \sigma = 0.25,\ T = 3,\ r = 3\%\ pa,\ \nu = 29$

Delta = ?

(iii) Delta-hedged replicating portfolio:-

$\Delta = 0.801$ shares

$\therefore$

$$C - \Delta^* S_0 = 17.91 - 0.801^* 60$$
$$= \boxed{\$30.15} \text{ short in Cash}$$

(iv) Using approximation [$\because$ better & less time - Consuming than BSM] :-

$$f(S, \sigma + \delta) \approx f(S, \sigma) + \delta \frac{df}{d\sigma}$$

$\therefore$ We obtain an approximate option price of $\rightarrow$
$17.91 + 29 * 0.02 =$ **\$18.49**

Q. 8

(i) Delta is nothing but a Greek notation measure which computes the Change in Option Price by the Change in Underlying asset i.e:-

$$\Delta = \frac{df}{dS} = \frac{df}{dS}(t, S_t)$$

(ii) Given data:-

EC $\{ S_0 = 100,\ r = 3\%,\ X = 109.42,\ T = 1,\ \Delta / N(d_1) = 0.42074$

Volatility = ?

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$\rightarrow$

$$\Phi(0.42079) = \frac{\ln\left(\frac{100}{109.42}\right) + \left(0.03 + 0.5(\sigma)^2\right)}{\sigma}$$

$$0.66276\sigma = -0.09 + 0.03 + 0.5\sigma^2$$

$$\therefore \underbrace{0.5\sigma^2}_{a} \underbrace{- 0.66276\sigma}_{b} \underbrace{- 0.06}_{c} = 0$$

Using quadratic formula method:-

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{+0.66276 \pm \sqrt{(-0.66276)^2 + 4(0.5 \times +0.06)}}{2(0.5)}$$

$$= -0.0851 \text{ or } \boxed{1.411}$$

$\boxed{\sigma = 1.411\%}$ [Tallied via Table f^n as well]

Q.9

(i) The f^n g must satisfy:

$$\frac{dg}{dt} + (r - q)S_t \frac{dg}{dS_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{d^2 g}{dS_t^2} = rg$$

[Taking the assump^n the above is a dividend paying stock]

r: risk-free rate

q: compounded-dividend rate

σ: assumed volatility from time "0" to "T"

∴ Boundary Condition:

$$D_T = g(T, S_T) = f(S_T)$$

(ii) ~~Rule~~ From part (i) we know that,

$$\frac{dg}{dt} = -\mu \frac{S_t^{n}}{S_0^{n-1}} e^{\mu(T-t)} = -\mu g$$

$$\frac{dg}{dS_t} = \frac{n S_t^{n-1}}{S_0^{n-1}} e^{\mu(T-t)} = \frac{n}{S_t} g$$

$$\frac{d^2 g}{dS_t^2} = \frac{n(n-1) S_t^{n-2}}{S_0^{n-1}} e^{\mu(T-t)} = \frac{n(n-1)}{S_t^2} g$$

~~Subst~~ Substitute above into PDE for:

$$-\mu g + (r-q) S_t \cdot \frac{n}{S_t} g + \frac{1}{2}\sigma^2 S_t^2 \frac{n(n-1)}{S_t^2} g = rg$$

$$\therefore -\mu + (r-q)n + \frac{1}{2}\sigma^2 n(n-1) = r$$

Now, if we set $q=0$, we get:

$$\mu = rn - r + \frac{1}{2}\sigma^2 n(n-1)$$

$$= r(n-1) + \frac{1}{2}\sigma^2 n(n-1)$$

$$\boxed{\mu = \left(r + \frac{1}{2}\sigma^2 n\right)(n-1)}$$

Q.10

(i) Put-Call Parity for a Stock Paying No-dividend.

$\therefore \; c + Ke^{-r(T-t)} - P_t = S_t$

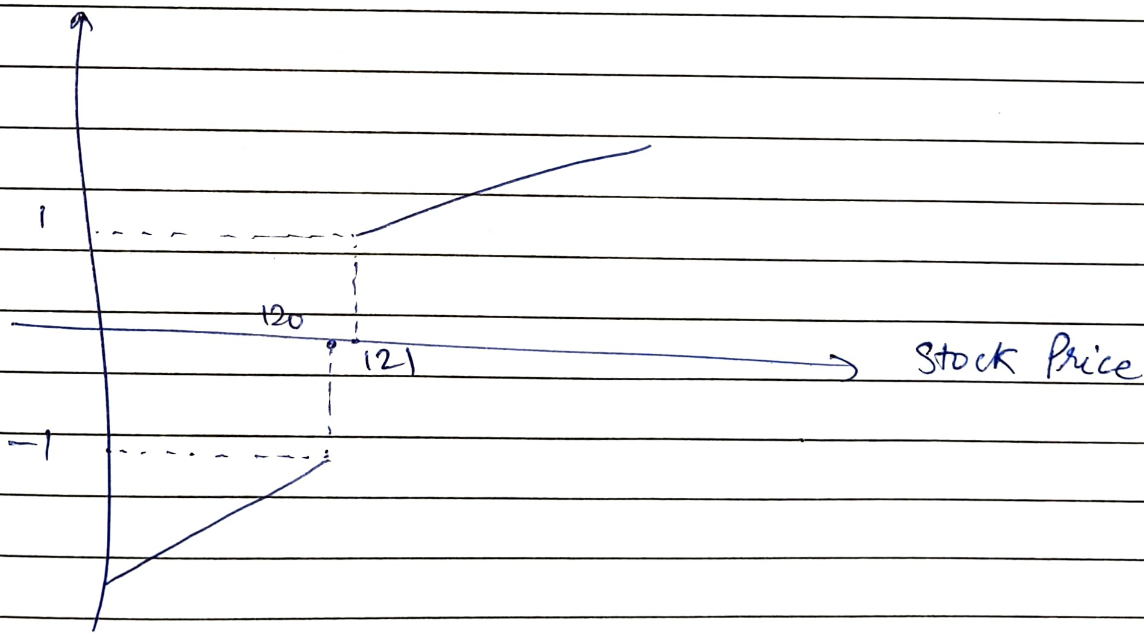
(ii) $S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2)$

$\because \; S_0 = 110, \; K = 120, \; r = 0.02, \; T = 1$

$\therefore \; d_1 = \dfrac{\ln\left(\frac{110}{120}\right) + (0.02 + 0.5 \times \sigma^2)}{\sigma}$

$d_2 = d_1 - \sigma$

(iii) Payoff



(iv) $S_1 - 121 \leq D \leq S_1 - 120$

$\therefore \; S_0 - 121e^{-r} \leq V \leq S_0 - 120e^{-r}$