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Probability & Statistics - 1

Assignment - 1

1. Let X be the amount of a home insurance claim and Y the amount of a car insurance claim.

$$X \sim N(800, 100^2) \qquad Y \sim N(1200, 300^2)$$

we require:

$$P((Y_1 + Y_2 + Y_3) > (X_1 + X_2 + X_3 + X_4) + 800)$$

$$= P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$$

we need distribution of $(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4)$:

$$= \sim N(3 \times 1200 - 4 \times 800,\ 3 \times 300^2 + 4 \times 100^2)$$

$$\sim N(400, 310000)$$

$$\Rightarrow P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$$

$$= P(Z > 0.718)$$

$$= 1 - P(Z < 0.718) = 0.236$$

3. (i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$, $x \geq \alpha$ & $\lambda, \alpha > 0$.

using the definition of an MGF:

$$M_X(t) = E[e^{tX}] = \int_\alpha^\infty e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$$

$$= \int_\alpha^\infty \lambda e^{\lambda\alpha} e^{-(\lambda-t)x} dx$$

$$= \left[\frac{-\lambda e^{\lambda\alpha}}{\lambda - t} e^{-(\lambda-t)x} \right]_\alpha^\infty$$

$$= \frac{\lambda}{\lambda - t} e^{t\alpha}$$ provided $t < \lambda$.

(ii) $$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$M'_X(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-2} e^{t\alpha} + \alpha\left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$\Rightarrow E(X) = M'_X(0) = \frac{1}{\lambda} + \alpha$$

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$$M''_x(t) = \frac{2}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-3} e^{t\alpha} + \frac{2\alpha}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2}$$

$$+ \alpha^2\left(1-\frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$\Rightarrow E(X^2) = M''_x(0) = \frac{2}{\lambda^2} + \frac{2}{\lambda}\alpha + \alpha^2$$

$$var(X) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left(\frac{1}{\lambda}+\alpha\right)$$

$$var(X) = \frac{1}{\lambda^2}$$

4. Probability, $P(X=n) = \frac{1}{n!} G^{(n)}_X(0)$

$$\rightarrow P(V=4) = \frac{1}{4!} G^4_V(0)$$

finding $G^4 v(t)$:

$$G'v(t) = kp^k q(1-qx)^{-k-1}$$

$$G''v(t) = -(-k-1)kp^k q^2(1-qx)^{-k-2}$$

$$G'''v(t) = (-k-2)(-k-1)kp^k q^3(1-qx)^{-k-3}$$

$$G^4 v(t) = -(-k-3)(-k-2)(-k-1)kp^k q^4(1-qx)^{-k-4}$$

Spiral

$$G_V^{4}(q)(0) = -(-k-3)(-k-2)(-k-1)kp^k q^4.$$

4. $f(x,y) = ax(x+y) \quad 0 < x < 5,$

we know that, $\quad 10 < y < 20$

$$\int_{10}^{20}\int_{0}^{5} ax(x+y)\,dx\,dy = 1$$

$$= \int_{10}^{20}\int_{0}^{5} a(x^2+xy)\,dx\,dy$$

$$= \int_{10}^{20} a\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5 dy = 1$$

$$= \int_{10}^{20} a\left[\frac{125}{3} + \frac{25}{2}y\right] dy = 1$$

$$= a\left[\frac{125}{3}y + \frac{25}{4}y^2\right]_{10}^{20} = 1$$

$$= a\left[\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right] = 1$$

$$= \cancel{a\,(833.33 \quad 3333.3333 =}$$

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$a(2\,291.666) = 1$

$a = 0.00044$

$$E(Y/x) = \int_y y \frac{f(x,y)}{f(x)} dy$$

$$f(x) = \int_y f(x,y)\,dy$$

$$= \int_{10}^{20} a(x^2 + xy)\,dy$$

$$= a\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= a\left[20x^2 + 200x - 10x^2 - 50x\right]$$

$$= a\left[10x^2 + 150x\right]$$

$$E(Y/x) = \int_{10}^{20} y \frac{\cancel{a}[x^2 + xy]}{\cancel{a}[10x^2 + 150x]} dy$$

$$= \int_{10}^{20} y \frac{(x + y)}{(10x + 150)} dy$$

$$= \frac{1}{10x + 150}\left[\frac{y^2}{2}x + \frac{y^3}{3}\right]_{10}^{20}$$

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$$= \frac{1}{10x+150}\left(200x + \frac{28000}{3} - 50x - \frac{1000}{3}\right)$$

$$= \frac{1}{10x+150}\left(150x + \frac{7000}{3}\right).$$

6. Examining Arthur's statements :-

$X \sim Poi(\lambda)$ & $Y \sim Poi(\mu)$

let $\cancel{Z = X - Y}$
$\Rightarrow \cancel{Y = X - Z}$

X has MGF
$M_X(t) = e^{\lambda(e^t - 1)}$

and
Y has MGF
$M_Y(t) = e^{\mu(e^t-1)}$

so the difference

X − Y has MGF :
$\cancel{M_X(t) M}$

$$M_X(t) M_{-Y}(t) = e^{\lambda(e^t-1)} e^{-\mu(e^t-1)}$$

$$= e^{(\lambda-\mu)(e^t-1)}$$

$\Rightarrow$ which is the moment generating function of a Poisson with parameter $\lambda - \mu$

$\rightarrow \quad X - Y \sim Poi(\lambda - \mu)$

so Arthur's statement is true.

similarly, examining Christine's statement:

$X \sim Poi(\lambda) \quad \& \quad Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t - 1)} \quad \& \quad M_Y(t) = e^{\mu(e^t-1)}$

so the sum $X + Y$ has MGF:

$$M_X(t) M_Y(t) = e^{\lambda(e^t - 1)} e^{\mu(e^t - 1)}$$

$$= e^{(\lambda + \mu)(e^t - 1)}$$

$\Rightarrow$ MGF of Poisson with parameter $\lambda + \mu$

$\Rightarrow \quad X + Y \sim Poi(\lambda + \mu)$

so Christine's statement is also true.

7. (i) $var\left(X/_{Y=2}\right) = E\left(X^2/_{Y=2}\right) - E^2\left(X/_{Y=2}\right)$

$$E\left(X/_{Y=2}\right) = 1 \times \frac{0}{0.6} + 2 \times \frac{0.3}{0.6} + 3 \times \frac{0.1}{0.6} + 4 \times \frac{0.2}{0.6}$$

$$= \frac{17}{6}$$

$$E\left(X^2/_{Y=2}\right) = 1^2 \times \frac{0}{0.6} + 2^2 \times \frac{0.3}{0.6} + 3^2 \times \frac{0.1}{0.6} + 4^2 \times \frac{0.2}{0.6}$$

$$= \frac{53}{6}$$

$$var\left(X/_{Y=2}\right) = \frac{53}{6} - \left(\frac{17}{6}\right)^2$$

$$= \frac{29}{36} \doteq 0.80556$$

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(ii) conditional expectation

$$E(U/V=v) = \int_u u\, f(u/v)\, du$$

$$\cancel{f(v) = \int_{u=0}^{1} \frac{48}{67}(2uv^2 \; du}$$

$$f(v) = \int_{u=0}^{1} \frac{48}{67}(2uv - u^2)\, du$$

$$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right]$$

$$\Rightarrow f(u/v) = \frac{f(u,v)}{f(v)} = \frac{\frac{48}{67}(2uv - u^2)}{\frac{48}{67}\left(v - \frac{1}{3}\right)}$$

$$= \frac{(2uv - v^2)}{\left(v - \frac{1}{3}\right)}$$

so :

$$E(U/V=v) = \int_0^1 \frac{2u^2 v - u^3}{v - \frac{1}{3}}\, du$$

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$$= \left[\frac{\frac{2}{3} u^3 v - \frac{u^4}{4}}{v - \frac{1}{3}} \right]_0^1$$

$$= \frac{\frac{2}{3} v - \frac{1}{4}}{v - \frac{1}{3}}$$

8. $X \sim \text{Gamma}(3, 2)$

$$E\left(Y/_{X = x}\right) = 3x + 1$$

$$\text{Var}\left(Y/_{X = x}\right) = 2x^2 + 5$$

$$E(Y) = E\left(E\left(Y/_X\right)\right) = E(3x + 1)$$

$$= 3E(x) + 1$$

but $E(x) = \frac{3}{2} = 1.5$

$$\Rightarrow \quad E(Y) = 3(1.5) + 1$$

$$= 4.5 + 1$$

$$= \underline{\underline{5.5}}$$

using $var(Y) = E[var(Y/X)] + var[E(Y/X)]$

$= var(Y) = E[2x^2 + 5] + var[3X + 1]$

$= 2E[X^2] + 5 + 9\, var(X)$

$= 2\left((1.5)^2 + \left(\frac{9}{16}\right)\right) + 9\left(\frac{3}{4}\right)$

$= 2[2.8125) + 6.75$

$\boxed{var(Y) = 12.375}$

9. $E[X] = 3$

$sd_x = 2$

$E[Y] = 4$

$sd_y = 1$

correlation coefficient $= -0.3$

$Z = X + Y$

(i) $cov(X, Z) = cov(X, X + Y)$

$= cov(X, X) + cov(X, Y)$

$= var(X) + cov(X, Y)$

using correlation coefficient between X & Y gives:

$$\text{corr}(X,Y) = -0.3 = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \frac{\text{cov}(X,Y)}{\sqrt{4 \times 1}}$$

$$\text{cov}(X,Y) = -0.6$$

$$\Rightarrow \text{cov}(X,Z) = 4 - 0.6 = \underline{\underline{3.4}}$$

(ii) $\text{var}(Z) = \text{cov}(Z,Z)$:

$$\text{var}(Z) = \text{cov}(X+Y, X+Y)$$

$$= \text{cov}(X,X) + 2\,\text{cov}(X,Y) + \text{cov}(Y,Y)$$
$$= \text{var}(X) + 2\,\text{cov}(X,Y) + \text{var}(Y)$$
$$= 4 + 2(\cancel{3.4}\ (-0.6)) + 1$$
$$= 3.8$$

10. $$k x^{-\alpha} e^{-\frac{y}{\beta}} \qquad 1 < x < \infty$$
$$1 < y < \infty$$

since the pdf must integrate to 1

$$\int_{y=1}^{\infty}\int_{x=1}^{\infty} k x^{-\alpha} e^{y/\beta}\, dx\, dy = 1$$

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integrating over x values give:

$$\int_1^{\infty} k x^{-\alpha} e^{-y/\beta} = k e^{-y/\beta}\left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^{\infty}$$

$$= \frac{k e^{-y/\beta}}{\alpha - 1}$$

integrating this over y values:

$$\int_1^{\infty} \frac{k e^{-y/\beta}}{\alpha - 1} = \frac{k}{\alpha - 1}\left[-\beta e^{-y/\beta}\right]_1^{\infty}$$

$$\frac{k \beta e^{-1/\beta}}{\alpha - 1}$$

equating this to 1:

$$\frac{k \beta e^{-1/\beta}}{\alpha - 1} = 1$$

$$\boxed{k = \frac{(\alpha - 1) e^{1/\beta}}{\beta}}$$