Unit 3 & 4 Priyanshu

SRM-2 Assignment-2 $\boxed{432}$

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~~(iv) We know that,~~

~~$\pi = \pi P$~~

~~The Eqns~~

Q.1

Given:-

Bid → offer (λ)

offer → Bid (μ)

(i)

$$\begin{array}{c|cc} & B & O \\ \hline B & -\lambda & \lambda \\ O & \mu & -\mu \end{array}$$

(ii) For Bid:-

$B \sim exp(\lambda_i)$

For offer:-

$O \sim exp(\mu_i)$

(iii) Kolmogorov Forward Eqs:-

We know that →

$$\frac{d P_{ij}(t)}{dt} = \sum_{k \in S \cdot S} P_{ik}(t) \cdot \mu_{kj}$$

∴ $B \to B$

$$\frac{d\, {}_tP_s^{BB}}{dt} = {}_tP_s^{Bo}(\mu) + {}_tP_s^{BB}(-\lambda)$$

$B \to 0$

$\therefore \frac{d\, {}_tP^{B0}_s}{dt} = {}_tP^{BB}_s(\lambda) + {}_tP^{B0}_s(-\mu)$

Since we have 2 ~~sided~~ state model:

$\therefore {}_tP_s^{BB} + {}_tP^{B0}_s = 1$

Substituting the sol^n found in part (iii):

$\frac{d\, {}_tP_s^{BB}}{dt} = -\lambda\, {}_tP^{BB}_s + \mu\, {}_tP^{B0}_s$

Integrating factor $= e^{\int P(x) \cdot dx}$

$\cancel{\frac{d}{dt}\, e^{(\lambda+\mu)t} \cdot {}_tP_s^{BB}}$

$\frac{d\, {}_tP_s^{BB}}{dt} = -\lambda\, {}_tP^{BB}_s + \mu(1 - {}_tP^{BB}_s)$

$= -\lambda\, {}_tP^{BB}_s + \mu - \mu\, {}_tP^{BB}_s$

$\frac{d\, {}_tP_s^{BB}}{dt} = -{}_tP^{BB}_s(\lambda+\mu) + \mu$

[Arranging in IF terms]

$\frac{d\, {}_tP^{BB}_s}{dt} + {}_tP^{BB}_s(\lambda+\mu) = \mu$ — ①

$\therefore e^{\int \lambda+\mu \cdot dt}$

[On solving & putting back in ①, we get]

$\frac{d\, e^{(\lambda+\mu)t} \cdot {}_tP_s^{BB}}{dt} = \mu \cdot e^{(\lambda+\mu)t}$

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Next,

$$e^{(\lambda+\mu)t} \cdot {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} \cdot e^{(\lambda+\mu)t} + \text{Constant} \quad \text{—} \; (2)$$

$\because$ process stays in the same state i.e Bid at time s,

${}_tP_s^{BB} = 1$ and so constant after simplifying (2) is :-

$$\frac{\lambda}{\mu+\lambda}$$

$$\therefore \boxed{{}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{(-(\lambda+\mu)t)}}$$

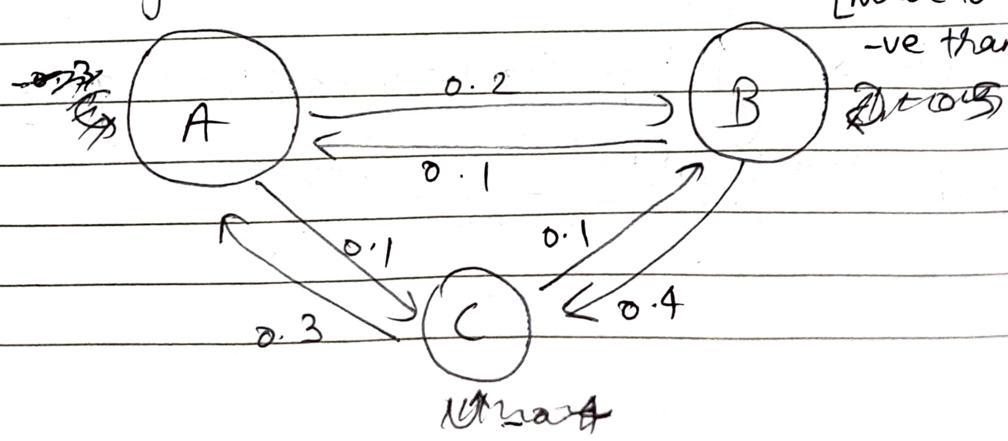
Q.2

(i) Given :-

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & -0.3 & 0.2 & 0.1 \\ B & 0.1 & -0.5 & 0.4 \\ C & 0.3 & 0.1 & -0.4 \end{array}$$

Transition Graph :-

[Never to mention -ve transitions]



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(ii) We need
KFE for:-

a] $\frac{d\ P_{AA}(t)}{dt} = {}_tP^{AB}(t)(0.1) + {}_tP^{AA}(t).(-0.3) + {}_tP^{AC}(t)(0.3)$

b] $\frac{d\ P_{AB}(t)}{dt} = {}_tP^{AB}(t)(-0.5) + {}_tP^{AA}(t)(0.2) + {}_tP^{AC}(t)(0.1)$

c] $\frac{d\ P_{AC}(t)}{dt} = {}_tP^{AC}(t)(-0.4) + {}_tP^{AB}(t)(0.4) + {}_tP^{AA}(t)(0.1)$

(iii) P[Staying in State A from time 0 to 2] = Probability of holding time.

$\therefore e^{-\int_0^2 \lambda dt}$

$= e^{-\int_0^2 0.3 dt}$

$= e^{-0.3\times 2 + 0}$

$= \boxed{e^{-0.6}}$

(iv) In the given Qs the only possible third jumps are:-
B→A→C, C→A→C & C→B→C.

∵ the probab of each jump is given by ratio of transⁿ rates, probability of each path is as follows:-

$CBC = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} = \boxed{\frac{1}{15}}$

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$BAC = \frac{2}{3} \cdot \frac{1}{5} \cdot \frac{1}{3} = \boxed{\frac{2}{45}}$; $CAC = \frac{1}{3} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{\cancel{3}} = \boxed{\frac{1}{12}}$

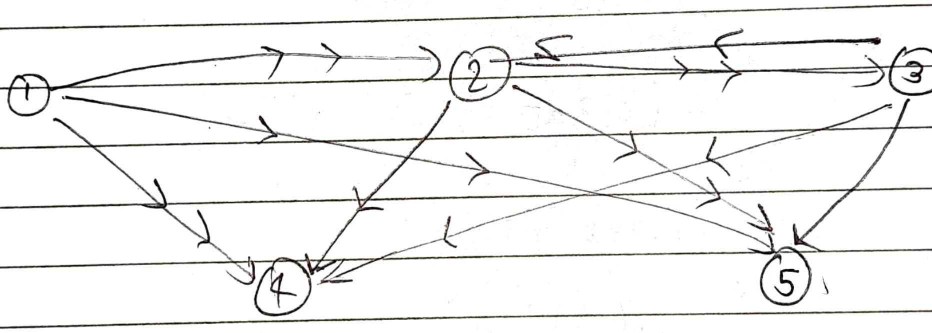
Addition : $\frac{2}{45} + \frac{1}{12} + \frac{1}{15} = \frac{7}{36} = \boxed{0.194}$

↳ Probab of having third jump into C.

Q.3

(i) Given:-

Never taken Nimble ①
Taking Nimble ②
No longer taking Nimble ③
Dead thru heart disease ④
Dead thru other cause ⑤



(ii) S.T.

$$\frac{d}{dt}\left({}_tP_x^{(3,4)}\right) = {}_tP_x^{32}\,\mu_{x+t}^{24} + {}_tP_x^{33}\,\mu_{x+t}^{34}$$

The possible formations with the help of Chapman Kolmogorov Eqs are as follows:-

$${}_{dt+t}P_x^{34} = {}_tP_x^{31} \cdot {}_{dt}P_{x+t}^{14} + \underline{{}_tP_x^{32}\,{}_{dt}P_{x+t}^{24}} + \underline{{}_tP_x^{33}\,{}_{dt}P_{x+t}^{34}}$$
$$+ \underline{{}_tP_x^{34}\,{}_{dt}P_{x+t}^{44}} + {}_tP_x^{35}\,{}_{dt}P_{x+t}^{54}$$

However since $3 \to 1$ and $5 \to 4$ jumps aren't possible.

$$\therefore \; {}_{dt+t}p_x^{34} = {}_tp_x^{32} \cdot {}_{dt}p_{x+t}^{24} + {}_tp_x^{33} \cdot {}_{dt}p_{x+t}^{34} + {}_tp_x^{34} \cdot {}_{dt}p_{x+t}^{44} \quad - (1)$$

From part (i) we know that ${}_{dt}p_{x+t}^{44} = 1$

We know for small dt that:-

$${}_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij} \cdot dt + o(dt) \dots i \neq j$$

Using the same concept in (1):-

$$dt + {}_tp_x^{34} = {}_tp_x^{32} \cdot \mu_{x+t}^{24} + {}_tp_x^{33} \cdot \mu_{x+t}^{34} + {}_tp_x^{34} \quad \cancel{+\,{}_tp_x^{34}\,\mu_{x+t}^{44}} \quad o(dt)$$

$$\frac{{}_{dt+t}p_x^{34} - {}_tp_x^{34}}{} = {}_tp_x^{32} \cdot \mu_{x+t}^{24} + {}_tp_x^{33} \cdot \mu_{x+t}^{34}$$

$$\frac{d}{dt}({}_tp_x^{34})$$

Thus SHOWED

Q.4

(i) Since the mean = parameter, there are total of 3 calls per hour.

(ii) The process is memoryless so the fact that Fred hasn't had a call for 15 mins stands irrelevant.

Expected next call time is 20 mins.

(iii) P[Zero calls in time 0.5 hrs]

Using $P_j(t) = \frac{e^{-\lambda t}(\lambda t)^j}{j!}$

$\therefore P_0(0.5) = e^{-1.5} = \boxed{0.2231}$

(iv) The expected time that Fred is on the phone is expected no. of calls times the expected length of call.

Per hour: 3 calls × 7 mins = 21 mins
So, probab that phone is engaged is $\frac{21}{60} = 0.35$

Q.5

(i) Using Markov Property, we have:-

$$P_{ii}(t, t+dt) = 1 - \sum_{j \neq i} \lambda_{ij}(t)\,dt + o(dt)$$

where the λs are the instant trans^n rates and $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then substituting, we have

$P_{HH}(x, t+dt) = P_{HH}(x,t)(1 - \sigma(t)dt - \mu(t)dt + P_{HS}(x,t) f(t, C(t)) + o(dt)$

$\therefore P_{HH}(x, t+dt) - P_{HH}(x,t) = P_{HH}(x,t)(-\sigma(t) - \mu(t)) \cdot dt + P_{HS}(x,t) f(t, C(t))\,dt + o(dt)$

$\therefore \frac{d}{dt} P_{HH}(x,t) = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x,t)}{dt}$

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$$= P_{HH}(x,t)\left(-\sigma(t)-\mu(t)\right) + P_{HS}(x,t)\,\rho(t, C(t))$$

(ii) The eq^n simplifies when considering $P_{\overline{HH}}(t)$ to

$$\frac{d}{dt} P_{\overline{HH}}(0,t) = -\left(\sigma(t)+\mu(t)\right) P_{\overline{HH}}(t)$$

$$= \frac{1}{P_{\overline{HH}}(0,t)} \cdot \frac{d}{dt} P_{\overline{HH}}(0,t) = -\left(\sigma(t)+\mu(t)\right) = \frac{d}{dt} \ln P_{\overline{HH}}(t)$$

Integrating both sides:

$$\left[\ln P_{\overline{HH}}(0,t)\right]_0^t = \int_{s=0}^{t} -\left(\sigma(s)+\mu(s)\right) \cdot ds$$

$\therefore$ as $P_{\overline{HH}}(0) = 1$

$$P_{\overline{HH}}(0,t) = e^{-\left(\int_{s=0}^{t} (\sigma(s)+\mu(s)) \cdot ds\right)}$$

Q.7

(i) The MLE of the trans^n intensity from State i to j is the no. of transitions from state i to j, divided by the total waiting time in State i.

To estimate the trans^n intensities exactly we therefore need the total time spent in each state.

(ii) Integrated forward eqs:-

$$P_{AA}(s,t) = \exp\left[-\int_{u=s}^{t} \mu \cdot du\right]$$

$$P_{AT}(s,t) = \int_{u=0}^{t} P_{AA}(s,u)\, \mu\, 1\, du$$

(iii) Measure from time zero i.e, $s=0$ and drop s from notation

Thus,
Solving the integrated KFE :-

$$P_{AT}(T) = \int_{s=0}^{T} e^{(-\mu s)} \mu \, ds = \left[-e^{(\mu s)}\right]_0^T = 1 - e^{(-\mu T)}$$

Hence, Expected Cost is $C(1-\exp(-\mu T))$

(iv)

$$\begin{tikzcd}
\boxed{\text{Active Policy}} \arrow[r, "\mu"] \arrow[d, "\lambda"] & \boxed{\text{Theft claim}} \\
\boxed{\text{Lapsed}} &
\end{tikzcd}$$

(v) We now have $\frac{d}{dt} P_{AA}(t) = -P_{AA}(t)(\mu+\lambda)$

So $P_{AA}(t) = e^{(-(\mu+\lambda)t)}$

We want $\frac{d}{dt} P_{AT}(T) = P_{AA}(t) \, \mu = \mu \exp^{(-(\mu+\lambda)t)}$

Solving this produces $P_{AT}(t)$:

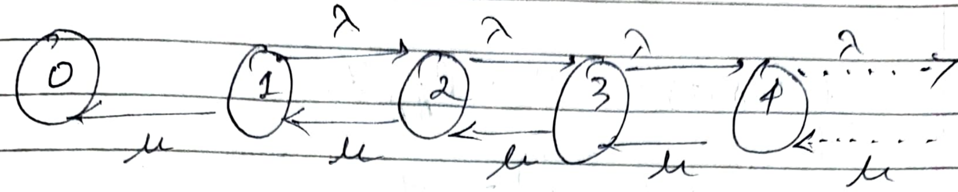
$$\left| \frac{-\mu}{(\mu+\lambda)} \exp(-(\mu+\lambda)t)\right|_0^T = \frac{\mu}{\mu+\lambda}\left(1-\exp(-(\mu+\lambda)T)\right)$$

So claims become $\frac{\mu}{\mu+\lambda} C(1-\exp(-(\mu+\lambda)T))$

8.

i) $\{0, 1, 2, 3, 4, \ldots\}$

ii)



iii) Generator matrix

$$\begin{array}{c} \text{lives} \quad 0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \quad \ldots \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) \end{pmatrix} \end{array}$$

iv) EITHER

If a Markov jump process X_t is examined at the times of transition, the resulting process is called the jump chain associated with X_t

Or

A jump chain is each distinct state visited in the order visited where the time set is the times when states are moved between.

v) Lives 0 1 2 3 4 ...

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 \\ 0 & \mu(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 \\ 0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) \\ 0 & 0 & 0 & \mu/(\mu+\lambda) & 0 \\ \text{etc.} & & & & \end{pmatrix} \text{etc.}$$

vi) $\left[\frac{\mu}{\mu+\lambda}\right]^3$

Q10 i)

$$\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -2t \times P_{\bar{A}\bar{A}}(t)$$

$$= \frac{d}{dt}\left[\ln P_{\bar{A}\bar{A}}(t)\right] = -2t$$

$$= \ln P_{\bar{A}\bar{A}}(s) = -s^2 + \text{constant}$$

We also know $P_{\bar{A}\bar{A}}(0) = 1$, hence constant $= 0$

Hence, $P_{\bar{A}\bar{A}}(s) = e^{-s^2}$

(ii) $P = \int_0^T P(\text{remains in A to time s}) \times P(\text{trans}^n \text{ to B in time } s, s+ds) \times P(\text{remains in B to time T})\, ds$

$$= \int_{s=0}^{T} P_{\bar{A}\bar{A}}(s) \times 2s \times P_{\bar{B}\bar{B}}(s,T)\, ds$$

Using result from part (i) and similar result for P_{BB} with boundary condition $P_{BB}(s,s) = 1$, this gives us:

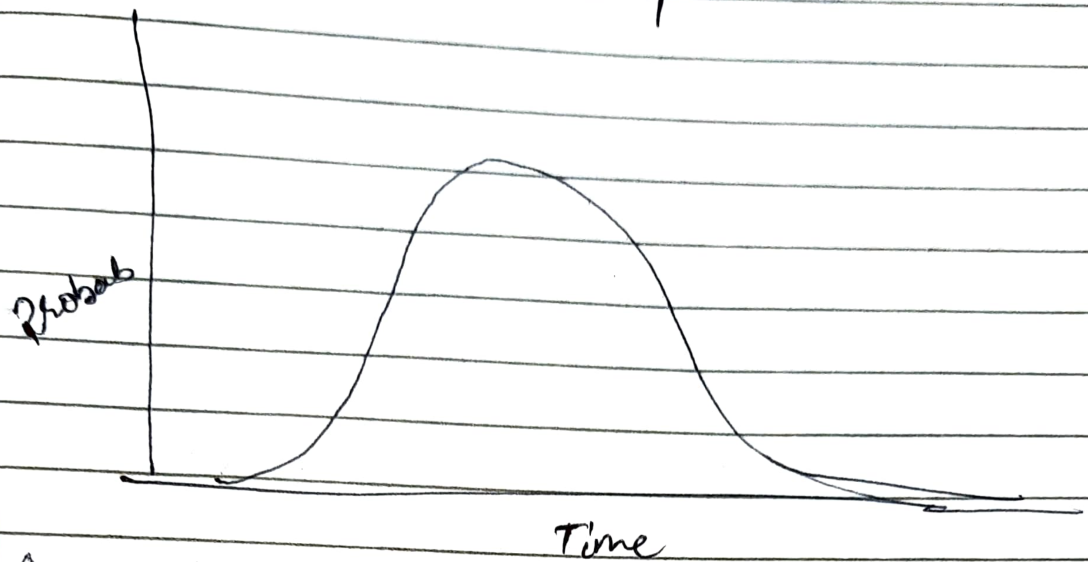
$$= \int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2} \cdot ds$$

$$= \int_{s=0}^{T} 2s \times e^{-T^2}\, ds$$

$$= e^{-T^2} \times T^2$$

(iii) The sketch should be shaped like

a]



b] Comment:

→ Initial probab increases from 0 at T=0, then accelerates as transⁿ rate from A to B increases.

→ However, as transⁿ increase, it becomes more likely that process has already visited state B and jumped back to A.

∴ Probab of being in first visit to B tends (exp) to zero.

c] differentiate to find turning point: