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Probability and Statistics 2
Unit 1 and 2
Assignment Questions

1. Given: $X \sim Poisson(\theta)$

$\frac{X-\theta}{\sqrt{\theta}} \sim$ approx. $N(0,1)$

$n=1, \quad \hat{\lambda}=200$

Here, $\frac{\hat{\lambda}-\lambda}{\sqrt{\lambda/n}} \sim N(0,1)$

$\hat{\lambda} = \frac{\sum X_i}{n}$

Hence, $P(Z_{2.5\%} < Z < Z_{97.5\%}) = 0.95$

$P(-1.96 < Z < 1.96) = 0.95$

$P\left(-1.96 < \frac{\hat{\lambda}-\lambda}{\sqrt{\lambda/n}} < 1.96\right) = 0.95$

$P\left(-1.96\sqrt{\frac{\hat{\lambda}}{n}} - \hat{\lambda} < \lambda < 1.96\sqrt{\frac{\hat{\lambda}}{n}} - \hat{\lambda}\right) = 0.95$

$P\left(\hat{\lambda} - 1.96\sqrt{\frac{\hat{\lambda}}{n}} < \lambda < \hat{\lambda} + 1.96\sqrt{\frac{\hat{\lambda}}{n}}\right) = 0.95$

$\hat{\lambda} = 200, \; n=1$

95% CI for $\lambda = \left(200 - 1.96\sqrt{\frac{200}{1}}, \; 200 + 1.96\sqrt{\frac{200}{1}}\right)$

$\Rightarrow (172.2814142, \; 227.7185858)$

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2. $X_i : i = 1,2,\dots,n$ iid's
mean $\to \mu$, var. $\to \sigma^2$
$\bar{X} = \sum_1^n X_i$, $S^2 = \frac{1}{n-1} \sum_1^n (X_i - \bar{X})^2$

i) To show $\to$ $E(\bar{X}) = \mu$, $V(\bar{X}) = \frac{\sigma^2}{n}$

$E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right) = \frac{\sum E(X_i)}{n} = \frac{n\mu}{n} \Rightarrow \boxed{\mu}$

$V(\bar{X}) = V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2} V(\sum X_i) = \frac{1}{n^2} \sum V(X_i) = \frac{n\sigma^2}{n^2}$

$\Rightarrow \boxed{\frac{\sigma^2}{n}}$

ii) To show $\to$ $E(S^2) = \sigma^2$ $X_i \sim$ Norm.
$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2$

$E(S^2) = E\left[\frac{1}{n-1}(\sum X_i^2 - n\bar{X}^2)\right] = \frac{1}{n-1}\left[E(\sum X_i^2) - nE(\bar{X}^2)\right]$

$= \frac{1}{n-1}\left[\sum E(X_i^2) - n E(\bar{X}^2)\right]$

Now, $Var(X_i) = E(X_i^2) - [E(X)]^2$
$E(X_i^2) = \mu^2 + \sigma^2$
$Var(\bar{X}) = E(\bar{X}^2) - [E(\bar{X})]^2$
$E(\bar{X}^2) = \mu^2 + \frac{\sigma^2}{n}$

$E(S^2) = \frac{1}{n-1}\left[\sum(\mu^2+\sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right]$

$= \frac{1}{n-1}(n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2) = \frac{1}{n-1} \times \sigma^2(n-1)$

$E(S^2) = \boxed{\sigma^2}$

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iii) ~~To~~ To show → $S^2 = \frac{2\sigma^4}{n-1}$

$X \sim N(\mu, \sigma^2)$

Then, $\frac{(n-1) S^2}{\sigma^2} \sim \chi^2_{n-1}$

$V\left[\frac{(n-1)S^2}{\sigma^2}\right] = \frac{(n-1)^2}{\sigma^4} . V(S^2)$

$V(\chi^2_{n-1}) = \frac{(n-1)^2}{\sigma^4} V(S^2)$

$2\cancel{(n-1)} = \frac{(n-1)\cancel{(n-1)}}{\sigma^4} V(S^2)$

$$\boxed{V(S^2) = \frac{2\sigma^4}{n-1}}$$

iv) $P(0.5\sigma^2 < S^2 < 1.5\sigma^2)$

$= P(S^2 < 1.5\sigma^2) - P(S^2 < 0.5\sigma^2)$

$n = 10$

$\sigma^2 = 100$

$P(S^2 < 1.5\sigma^2) = P(S^2 < 150) = P\left(\frac{9S^2}{\sigma^2} < \frac{150 \times 9}{100}\right)$

$= P(\chi^2_9 < 13.5) = 0.8587$

$P(S^2 < 0.5S^2) = P(S^2 < 50)$

$= P\left(\frac{9S^2}{\sigma^2} < \frac{50 \times 9}{100}\right) = P[\chi^2_9 < 4.5]$

$= 0.1245$

$P(50 < S^2 < 150) = 0.8587 - 0.1245$

$= 0.7342$

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3. i) The likelihood function is :

$$L(p) = C\,[(1-p)^4]^{86}\,[4p(1-p)^3]^{75}\,[6p^2(1-p)^2]^{16}\,[4p^3(1-p)]^2\,[p^4]^1$$

$$= C[(1-p)^{344}][p^{75}(1-p)^{225}][p^{32}(1-p)^{32}][p^6(1-p)^2][p^4]$$

$$= C(1-p)^{603}\,p^{117} \quad \text{, where } C \text{ is constant.}$$

Taking logs and differentiating with respect to p and setting equal to zero gives :

$$\frac{d}{dp}\ln L = \frac{-603}{(1-p)} + \frac{117}{p} = 0$$

$$p = \frac{117}{(603+117)} = \frac{117}{720} = 0.1625$$

Checking for maximum :

$$\frac{d^2}{dp^2}\ln L = \frac{-603}{(1-p)^2} \cdot \frac{-117}{p^2} < 0$$

Hence, maximum value for Ln L is at $p = 0.1625$.

ii) Goodness of fit test :

Testing hypotheses using χ^2 goodness of fit test :

H_0 the prob. conform to a Bin (4, p) dist.

H_1 the prob. do not conform to a Bin(4, p) dist.

Using $\hat{p} = 0.1625$ from part (a), the prob. for Binomial dist. are :

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$P(X=0) = (1-p)^4 = 0.49197$
$P(X=1) = 4p(1-p)^3 = 0.38183$
$P(X=2) = 6p^2(1-p)^2 = 0.11113$
$P(X=3) = 4p^3(1-p) = 0.01437$
$P(X=4) = p^4 = 0.00070$

Expected values → 88.55, 68.73, 20.00, 2.59, 0.13
Combining Expected Values <5 to third group,

No. of Claims	0	1	2 or more
Observed O_i	86	75	19
Expected E_i	88.55	68.73	22.72

Degrees of freedom = 3-1-1=1

$$\chi^2 = \Sigma (O_i - E_i)^2/E_i$$
$$= \frac{(86-88.55)^2}{88.55} + \frac{(75-68.73)^2}{68.73} + \frac{(19-22.72)^2}{22.72}$$
$$= 0.074 + 0.572 + 0.608$$
$$= 1.254$$

This is less than the 5% critical value of 3.841.
We have insufficient evidence at 5% level to reject H_0.
Hence, the model is a good fit.

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4. $f(x) = \frac{c\beta^2}{(x+\beta)^4}$; $x>0,\ \beta>0$
where c is constant and
β is a parameter.

i) Using 2 parameter version of Pareto distribution

$PDF = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}$ Hence, β corresponds to α.

Thus, $c=3$ (as c corresponds to α here)

$E(X) = \frac{\lambda}{\alpha-1}$ $E(X) = \frac{\beta}{2}$

$V(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = \frac{3\beta^2}{4}$

ii) $E(X) = \frac{\beta}{2} = \overline{X}$

$\boxed{\hat{\beta} = 2\overline{X}}$

$$Bias = E(\hat{\beta}) - \beta = E(2\overline{X}) - \beta = 2E(\overline{X}) - \beta$$
$$= \frac{2}{n}\sum E(X_i) - \beta = \frac{\cancel{2}}{\cancel{n}} \times \frac{\cancel{n}\beta}{\cancel{2}} - \beta$$

$Bias = 0$, hence unbiased.

$$MSE = V(\hat{\beta}) + (Bias)^2 = V(\hat{\beta}) = V(2\overline{X}) = 4V(\overline{X})$$
$$= 4V\left(\frac{\sum X_i}{n}\right) = \frac{4}{n^2}\sum V(X_i) = \frac{\cancel{4}}{n^{\cancel{2}}} \times \frac{3\beta^2 \cancel{n}}{\cancel{4}}$$

$MSE = \frac{3\beta^2}{n}$

As n increases, MSE tends to 0.
Hence, $\hat{\beta}$ is constant.

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iii) $\hat{\beta} = b\bar{X}$

$$E(\hat{\beta}_n) = E(b\bar{X}) = bE\left(\frac{\sum X_i}{n}\right) = \frac{b}{n}\sum E(X_i)$$

$$= \frac{b}{\cancel{n}} \times \frac{\beta}{2} \times \cancel{n} = \frac{b\beta}{2}$$

$$V(\hat{\beta}_n) = V(b\bar{X}) = b^2 V\left(\frac{\sum X_i}{n}\right) = \frac{b^2}{n^2}\sum V(X_i)$$

$$= \frac{b^2}{n^2} \times \frac{3}{4}\beta^2 \times \cancel{n} = \frac{3}{4}\frac{b^2\beta^2}{n}$$

$$MSE = V(\beta_2) + Bias^2 = \frac{3}{4}\frac{b^2\beta^2}{n} + \left(\frac{b\beta}{2} - \beta\right)^2$$

$$= \frac{3b^2\beta^2}{4n} + \left(\frac{b\beta - 2\beta}{2}\right)^2 = \frac{3b^2\beta^2}{4n} + \left(\frac{b^2\beta^2 + 4\beta^2 - 4\beta^2 b}{4}\right)$$

$$= \frac{3b^2\beta^2 + nb^2\beta^2 + 4n\beta^2 - 4n\beta^2 b}{4n}$$

$$M = \frac{\beta^2}{4n}(3b^2 + nb^2 + 4n - 4nb)$$

Diff. $$\frac{dM}{db} = \frac{\beta^2}{4n}(6b + 2nb - 4n) = 0$$

$$6b + 2nb - 4n = 0.$$

$$3b + nb - 2n = 0$$

$$b(3+n) = 2n$$

$$b = \frac{2n}{3+n} = \frac{2}{\left(\frac{3+n}{n}\right)} \Rightarrow \boxed{\frac{2}{\left(1+\frac{3}{n}\right)}}$$

Checking maximum, $\frac{d^2M}{db^2} = \frac{\beta^2(6+2n)}{4n} > 0$

Hence d^2m/db^2 ~~minimises~~ is minimum

Thus, $b = \frac{2}{(1+3/n)}$ minimises the MSE.

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iv) $E(\hat{\beta}_2) = \frac{b\beta}{2} = \frac{2}{(1+3/n)} \times \frac{\beta}{2} = \frac{n\beta}{n+3}$

Bias $= E(\hat{\beta}_2) - \beta$

$= \frac{n\beta - \beta(n+3)}{n+3} = \frac{n\beta - n\beta - 3\beta}{n+3} = \frac{-3\beta}{n+3}$

$\therefore$, we have (-ve) bias.

It underestimates true parameter ~~of~~ β.

$MSE = \frac{\beta^2}{4n}(3b^2 + nb^2 + 4n - 4nb)$

$= \frac{\beta^2}{4n}(b^2(n+3) + 4n(1-b))$

$= \frac{\beta^2}{4n}\left[\frac{4n^2}{(n+3)^2} \times (n+3) + 4n\left[\frac{3+n-2n}{n+3}\right]\right]$

$= \beta^2\left[\frac{4n^2}{n+3} + 4n\frac{(3-n)}{n+3}\right]$

$\beta^2\left[\frac{4n^2 + 12n - 4n^2}{n+3}\right]$

$MSE = \frac{12n\beta^2}{n+3}$

As $n \to \infty$, MSE doesn't tend to 0.
Hence, It is inconsistent.

5. i) $X \sim U(a,b)$

$E(X) = \frac{a+b}{2}$ $\quad \bar{X} = \frac{a+b}{2}$ $\quad \underline{a = 2\bar{X} - b}$, $\quad$ — ①

$V(X) = \frac{(b-a)^2}{12} \Rightarrow S^2 = \frac{(b-a)^2}{12}$, $12S^2 = (b-a)^2$, $2\sqrt{3}S = b-a$

$2\sqrt{3}S = 2\bar{X} - a - a$ $\quad$ {as $b = 2\bar{X} - a$)

$2\sqrt{3}S = 2\bar{X} - 2a$

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$\sqrt{3}\, S = \bar{X} - a$

$\hat{a} = \bar{X} - \sqrt{3}\, S$

ii) $b = 2\bar{X} - a$

$b = 2\bar{X} - \bar{X} + \sqrt{3} S$

$\hat{b} = \bar{X} + \sqrt{3} S$

iii) Sample of Observations : 1, 2, 8, 50, 150

$\bar{X} = 42.2$, $S = 63.57$

$\hat{a} = \bar{X} - \sqrt{3}S$

$= -67.906$

$\hat{b} = \bar{X} + \sqrt{3}\, S$

$= 152.3064$

Conclusion → • value of $\hat{b}$ is very close to sample value.
• value of $\hat{a}$ is very far from the true sample value, so it is the potential weakness of MOM.

iv) $X \sim U(0, b)$

$L = \prod_{x=1}^{n} f(x_i) = \left(\frac{1}{b}\right)^n$

$\log L = -n \log b$

$\frac{dL}{db} = \frac{-n}{b}$

$\frac{dL}{db} = 0 \Rightarrow \frac{-n}{b} = 0$ gives $b = \infty$ } not possible.

$\frac{d^2 l}{db^2} = \frac{n}{b^2} > 0$ minimum likelihood estimate.

Hence, $\hat{b} = \max(X_i)$

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6. $H_0 : p = 0.1$
$H_1 : p > 0.1$

i) P(type 1 error) = P(rejecting Ho / Ho is true)
Let X be the no. of hits in first 12 minute,
Y be no. of hits in step 12 minutes.

$P(\text{type 1 error}) = P(X \geq 3 | p = 0.1) + P(X = 0 / p = 0.1)$
$P(Y \geq 5 / p = 0.1) + P(X = 1 / p = 0.1)$
$P(Y \geq 4 | p = 0.1)$
$= (1 - 0.8891) + 0.2824(1 - 0.9957) + (0.6590 - 0.2824)$
$(1 - 0.9744) + (0.8891 - 0.6590)(1 - 0.8881)$
$= 0.14727 \approx 15\%$

ii) Probability of regressing null hypotheses when $p = 0.3$

$= P(X \geq 3 | p = 0.3) + P(X = 0 | p = 0.3) P(X \geq 5 | p = 0.3)$
$+ P(X = 1 / p = 0.3) P(X \geq 4 | p = 0.3) + P(X = 2 | p = 0.3)$
$P(Y \geq 3 | p = 0.3)$
$= (1 - 0.2528) + 0.0130(1 - 0.7237) + (0.0850 - 0.0138)$
$(1 - 0.4925) + (0.2528 - 0.0850)(1 - 0.2528)$
$= 0.9125 \approx 91\%$

iii) Probability of type II error
Accepting Ho when Ho is false
$= 1 - 0.91253$
$= 0.08747 \approx 9\%$ (where $p = 0.3$)

iv) 10 space agencies in aggregate fired 120 missiles and recorded Ho hits.

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$$\frac{x/n - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$n = 120$, $x = 40$

$\hat{p} = \frac{40}{120} = \frac{1}{3} = 0.3333$

$Z_{10\%} = 1.2816$

Lower bound of 90% right tailed confidence interval for 'p' is ÷

$$\hat{p} - 1.2816\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.3333 - 1.2816\sqrt{\frac{0.3333(1-0.3333)}{120}}$$
$$= 0.2782$$

v) $p > 0.278$ at 10% LOS.

$H_0: p = 0.278$ v/s $H_1: p > 0.278$

One sample Binomial Model

$\frac{X - np_0}{\sqrt{np_0q_0}} \sim N(0,1)$ with continuity correction.

$$\frac{39.5 - 120(0.278)}{\sqrt{120 \times 0.278 \times 0.722}} = 1.2511$$

- We don't have sufficient evidence to reject H_0, at 10% Level of Sig.

vi) Lower bound of Confidence Interval implies that is only a 10% chance of $p \leq 0.2782$ whereas from the Hypotheses test. 'p' could be less than or equal to 0.2780 with $P(x) > 10\%$.

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vii) $H_0: \mu_1 = \mu_2$ v/s $H_1: \mu_1 \neq \mu_2$

$$T.S = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

Given, $\bar{X}_1 = 65, \bar{X}_2 = 70, S_1 = 54, S_2 = 70, n_1 = 12, n_2 = 15$

$$S_p^2 = \frac{1}{25}\left[11(2916) + 14(4800)\right] = 4027.04$$

$$\frac{65-70}{65.45\sqrt{\frac{1}{12}+\frac{1}{15}}} = -0.2034$$

There is ± 2.060 $(\approx t_{25, 2.5\%})$
So we have insufficient evidences to reject H_0 at 5% level.

7. i) Public
$\bar{X}_1 = 30$
$S_1^2 = 64.3125$

Private
$\bar{X}_2 = 35.0555$
$S_2^2 = 67.4027$

ii) 2 sided t-test can be applied in case the samples come from populations with equal variances.
We are testing $H_0: \sigma_1^2 = \sigma_2^2$ v/s $H_1: \sigma_1^2 \neq \sigma_2^2$
Test Statistic is $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1+n_2-1}$

Value of statistics is $\frac{64.31}{67.42} = 0.9542$

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$F_{8,8}$ Values at 5% levels are 0.2256 & 4.433. Since the value of test statistics b/w the above values we have insufficient evidences to reject the hypotheses and conclude that the population variances are equal.

iii) $H_0 = \mu_{Br} = \mu_{Bu}$; $H_1 = \mu_{Bt} > \mu_{Bu}$

$$T.S. = \frac{(\bar{X}_2 - \bar{X}_1) - (\mu_2 - \mu_1)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{35.0555 - 30}{8.1152 \sqrt{\frac{2}{9}}}$$

T.S. = 1.32151

Tabulated Value $= t_{16, 5\%} = 1.746$

Since, $T.S_{cal} <$ Tabulated value, we have insufficient evidence to reject H_0.

8. i) Unbiased estimator → It is an estimator where expected value is equal to the parameter. Estimator is said to be unbiased if bias is equal to ~~true~~ zero for all values of parameter θ.

ii) Biased Estimator → It is an estimator which is consistently different from the parameter to be estimated.

iii) The mean square error of an estimator $\hat{\theta}$ is defined by $MSE(\hat{\theta}) = var(\hat{\theta}) + Bias^2$

iv) The estimator having less MSE is more consistent

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v) They both would be considered to be consistent when MSE minimises such that when, $n \to \infty$, $MSE \to 0$.

vi) i) Method of Moments
ii) Maximum Likelihood estimate

9. i) Variable t_K is defined as $t_K \Rightarrow \frac{N(0,1)}{\sqrt{\chi^2_K / K}}$, where K denotes degrees of freedom.

and the two random variables $N(0,1)$ and χ^2_K are independent.

ii) Mean of t_K for $K>2 = 0$
Variance of t_K for $K>2 = \frac{K}{K-2}$

iii) a) We know that for a sample from a normal population,
$$\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1}$$
For the given confidence level $t_9 = 3.25$,

Confidence Interval for μ is $\left(50 - 3.25 \times \sqrt{\frac{48.667}{10}},\ 50 + 3.25 \times \sqrt{\frac{48.667}{10}}\right)$

i.e. $(42.83, 57.17)$

b) From (ii) above, we know that $t_{n-1} \sim N\left(0, \sqrt{\frac{n-1}{n-3}}\right)$
$t_{n-1} \sim N\left(0, \sqrt{\frac{9}{7}}\right)$

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$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim N\left(0, \sqrt{\frac{9}{7}}\right)$$

$$\frac{\bar{X}-\mu}{S/\sqrt{7n/9}} \sim N(0,1)$$

Critical value for given level of confidence is 2.58.

Confidence interval for μ is $\left(50 - 2.58 \times \sqrt{\frac{\cancel{4.88}\ 48.667}{10} \times \frac{9}{7}}, \; 50 + 2.58 \times \sqrt{\frac{48.667}{10} \times \frac{9}{7}}\right)$

$\Rightarrow (43.54, 56.45)$

10. $n = 1000$ $\qquad X \sim Poi(n\lambda)$

$\lambda = 3$ $\qquad \Rightarrow X \sim Poi(3000)$

i) $P(\text{type 1 error}) = P(\text{reject } H_0 \,/\, H_0 \text{ is true})$

$= P(\cancel{\bar{x}}\ X > 3100 \mid X \sim Poi(3000))$

using normal approximation,

$P(X > 3100) = P\left(Z > \frac{3100 - 3000}{\sqrt{3000}}\right) = 1 - P(Z < 1.825)$

$= 3.39\,\%$

ii) Type II error = event of accepting the hypotheses when it is false.

iii) Power of list = P(regretting H_0 | H_0 is false)

$= P[X > 3100 \,/\, X \sim Poi(n\lambda) - N(n\lambda, n\lambda)]$

$= 1 - P\left[Z < \frac{3100 - n\lambda}{\sqrt{n\lambda}}\right]$

The value of power of test will depend on the value of parameter under hypotheses testing.

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iv) $\hat{\mu}$ is estimator of μ, $\hat{\mu}$ follows $N\left(\mu, \frac{\hat{\mu}}{n}\right)$

Hence, $\frac{\hat{\mu}-\mu}{\sqrt{\hat{\mu}/n}} \sim N(0,1)$, or

$P\left(-2.5758 < \frac{2.9-\mu}{\sqrt{2.9/1000}} < 2.5758\right) = 0.99$

Hence, the confidence Interval For μ is $(2.7613, 3.0387)$

11. $n=12$
$\Sigma X = 213,200$
$\Sigma X^2 = 4919,860000$
$\bar{X} = 17,767$
$S = 10,144$

New Sample Mean

$\Sigma X' = \Sigma X - 11000 - 48,000 + 27,500 + 31,500$
$= 213,200$
$\bar{X}' = \frac{213200}{12} = \cancel{17600}\ 17767.$

New Sample S.D.

$\Sigma X^{2\prime} = 4919860000 - (11000)^2 - (48000)^2 + (27500)^2 + (31500)^2$
$= 4243360,000$

$S'^2 = \frac{1}{n-1}\left(\Sigma X^{2\prime} - n(\bar{X}')^2\right) = \frac{1}{11}\left(4243360000 - 12(17767)^2\right)$
$= 41396775.64$

$S' = 6434.032611$

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12. i) The Cramer-Rao lower bound result holds under very general conditions except where the range of distⁿ under the parameter, such as the uniform distⁿ in this case.
This is due to a discontinuity, so the derivative in the Formula does not make sense.

ii) $Y = \max X_i$

Since, $0 < X_i < \theta$, Support of X will be 0 and θ.

For, $0 < y < \theta$

$$P(Y < y) = P(\max X_i < y)$$

$$= P\left[\prod_{i=1}^{n} (X_i < y)\right] = \prod_{i=1}^{n} P(X_i < y) \quad [\because X_i\text{s are independent}]$$

$$= \prod_{i=1}^{n} \left[\int_0^y \frac{dx}{\theta}\right]$$

$$= \left(\frac{y}{\theta}\right)^n$$

PDF of Y will be,

$$g_Y(y) = \frac{d}{dy} P(Y < y) = \frac{n}{\theta^n} y^{n-1} \quad \text{For } 0 < y < \theta$$

Now, For any non-negative real no. K,

$$E(Y^k) = \int_0^\theta y^k \frac{n}{\theta^n} y^{n-1}\, dy$$

$$= \frac{n}{n+k} \int_0^\theta \frac{n+k}{\theta^n} \cdot y^{n+k-1}\, dy = \frac{n}{n+k} \cdot \frac{\theta^{n+k} - 0}{\theta^n}$$

$$= \frac{n\theta^k}{n+k}$$

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iii) Bias of estimator $\hat{\theta}(c)$ is given as :

$$\text{Bias}(\hat{\theta}(c)) = E[\hat{\theta}(c) - \theta] = E[cY - \theta] = cE(Y) - \theta$$

$$= c\frac{n\theta}{n+1} - \theta$$ (using the results in (b) with $h=1$)

$$\left(\frac{cn}{n+1} - 1\right)\theta$$

Mean Square Error of the estimator $\hat{\theta}(c)$ is given as:

$$MSE[\hat{\theta}(c)] = E[\{\hat{\theta}(c) - \theta\}^2] = E[\{cY - \theta\}^2]$$

$$= E[c^2Y^2 - 2c\theta Y + \theta^2]$$

$$= c^2E[Y^2] - 2c\theta E(Y) + \theta^2$$

$$= c^2\frac{n\theta^2}{n+2} - c\frac{2n\theta^2}{n+1} + \theta^2$$ [using (ii) with $k=1, h=2$]

iv) For $\hat{\theta}(c)$ to be unbiased estimator of θ, we need,

$$\text{Bias}(\hat{\theta}(c_u)) = 0$$

$$\left(\frac{c_u n}{n+1} - 1\right)\theta = 0$$

$$c_u = \frac{n+1}{n}$$

v) In order to minimise $MSE[\hat{\theta}(c)]$, we need

$$0 = \frac{d}{dc} MSE[\hat{\theta}(c)]$$

$$= \frac{d}{dc}\left[c^2\frac{n\theta^2}{n+2} - \frac{c^2 n\theta^2}{n+1} + \theta^2\right] = 2c\frac{n\theta^2}{n+2} - \frac{2n\theta^2}{n+1}$$

$$c_m = \frac{2n\theta^2}{n+1} \cdot \frac{n+2}{2n\theta^2} = \frac{n+2}{n+1}$$

Note: $\frac{d^2}{dc^2} MSE[\hat{\theta}(c)] = \frac{d}{dc}\left[\frac{2cn\theta^2}{n+2} - \frac{2n\theta^2}{n+1}\right] = \frac{2n\theta^2}{n+2} > 0$, $\Rightarrow$ minimum

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vi) In order to minimise error in estimator, it is preffered to go For an estimator which has lower MSE among different competing estimators. So, here $\hat{\theta}(C_m)$ will be preffered over $\hat{\theta}(C_u)$.

As n becomes large, $\hat{\theta}(C_u) = \frac{n+1}{n} Y \to Y$

Similarly, $\hat{\theta}(C_m) = \frac{n+2}{n+1} Y \to Y$

Thus the two estimators becomes one and same as $n \to \infty$.