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ASSIGNMENT - 1

Q1 i) An opportunity through which investors gain from price mismatch of same underlying asset in 2 different markets is an 'Arbitrage opportunity'.

ii) Law of one price states that any 2 portfolios which have same weights should also have same price. If this does not holds then the investor can buy the underlying at cheap prices & sell it at a higher price to earn risk-free profit.

iii)
a) $K = 120$ (Put), $S_0 = 125$, $r = 0.05$, $t = 3$ months
Call $= 30$

$q = 15\%$ pa
Put call Parity,

$$p = c_i = Ke^{-r(T-t)} - S_t e^{-q(T-t)}$$
$$= 30 + 120e^{-0.05 \times 0.25} - 125e^{-0.15 \times 0.25}$$

$$p = 28.11$$

b) $p = 23$

If the price of put option is 23, the ~~can~~ can buy put at a cheaper price & sell call option at a higher price. Since put call parity doesn't hold true the ~~investor~~ investor will make arbitrage profit.

14 Dividends are denoted by:

$d_1, d_2, d_3, \ldots d_m.$

If the option price is exercised prior to the ex-divident date $= S_t - K$

If option is not exercised the price will drop to $S_t - d$.

The value of american option is greater than $S_t - d - Ke^{-r(T-t)}$

It is never optimal to exercise the option if

$S_t - d_n - Ke^{-r(T-t)} \geq S_{(t-n)} - K$

Using this,

$K(1-e^{-r(T-t)}) = 350(1-e^{-0.95(0.83-0.25)})$
$= 18.87$

and

$65(1-e-0.95(0.83-0.25)) = 10.91$

Q2) $dX_t = \alpha\mu(T-t)dt + \sigma\sqrt{(T-t)}\,dZ_t$

Let,

$A_t = \alpha\mu(T-t)$, $B_t = \sigma\sqrt{(T-t)}$ - 3

$$df = \frac{df}{dx}B_t\,dZ_t + \left(\frac{df}{dt} + \frac{df}{dx}A_t + \frac{1}{2}\frac{d^2f}{dx^2}B_t\right)$$

$$= -f\,B_t\,dZ_t + f\,\frac{dm}{dt}(T-t)dt - fA_t\,dt - \frac{1}{2}B_t^2\,dt$$

$$df = f\left(\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}B_t^2\right)dt - f\,B_t\,dZ_t$$

For f to be a martingale,

$$\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}B_t^2 = 0$$

$$\therefore \frac{dm}{dt}(T-t) = A_t + \frac{1}{2}Bt^2$$

$$\therefore \frac{dm}{dt}(T-t) = \alpha\mu - \frac{1}{2}\sigma^2 \text{ (from 2)}$$

Q3 i) $F(t,x) = e^{-t}x^2$

X is SDE

$$\frac{dX_t}{X_t} = 0.25\,dt + \sigma\,dW_t$$

$dY_t = ?$

$$\frac{df}{dt} = -e^{-t}x^2 = -f$$

$\frac{df}{dx} = e^{-t} 2x$

By Ito's Lemma,

$dy_t = \frac{df}{dt} \cdot dt + \frac{df}{dx} \cdot dx_t + \frac{1}{2} \frac{d^2f}{dx_2} \cdot \sigma^2 x_t^2 \, dt$

$= -f \cdot dt + 2e^{-t} x_t \, dx_t + e^{-t} \sigma^2 x_t^2 \cdot dt$

$= -y_2 \cdot dt + 2e^{-t} x_t^2 \cdot \frac{dx_t^2}{x_t} + e^{t} \sigma^2 x_t^2 \, dt$

$= -\frac{y}{2} \cdot dt + 2y + [0.25 dt + \sigma dW_t] + \sigma^2 y_t \, dt$

$\therefore \frac{dy_t}{y_t} = [2x - \frac{1}{4} + \sigma^2] dt + 2\sigma dW_t$

$= [\sigma^2 - 0.5] dt + 2\sigma dW_t$

$\therefore dy_t = [\sigma^2 - 0.5] y_t dt + 2\sigma y_t dW_t$

(ii) The process is martingale if drift is 0. which means $\sigma^2 - 0.5 = 0$, $\sigma^2 = 0.05$

Q5. Let S be the stock price

$P(S > 258)$

$\therefore S$ is a GBM

$S_t = S_0 \, e^{(\mu - \sigma^2/2)t + \sigma W_t}$

$\therefore \ln(St) \sim N((\ln(S0) + (\mu - \sigma^2/2)t + \sigma W_t), \sigma^2 t)$

$\therefore \ln(S_t) \sim \phi(\ln 254 + (0.16 - 0.35^2/2)0.5, 0.35 \times 0.5)$

$= \phi(5.59, 0.247)$

$\therefore P(S > 258) = 1 - N((5.55 - 5.59)/0.247)$

$= 1 - N(-0.1364)$

$= 0.5542$

The put option is exercised is the stock price less than Rs - 258 in 6 months time.
The probablity $= 1 - 0.5542$
$= 0.4457$

Q6. $\ln\left(\frac{S_t}{S_0}\right) = ut + \sigma B_t$

$\frac{S_t}{S_0} = e^{ut + \sigma B_t}$

$Y = S_t = S_0 e^{ut + \sigma B_t}$

(ii) $\frac{dS_t}{S_t} = x\,dB_t + y\,dt$

By ito's ~~lemma~~ lemma,

$dS_t = \frac{dy}{dt} dt + \frac{dy}{dB_t} dB_t + \frac{d^2y}{dB_t} \cdot \frac{dB_t^2}{2}$

$dS_t = uS_t dt + \sigma S_t dB_t + \frac{\sigma^2 S_t}{2} dt$

$= S_t \, dt\left(u + \frac{\sigma^2}{2}\right) + \sigma S_t \cdot dB_t$

$\frac{dS_t}{S_t} = x\,dB_t + y\,dt$

$x = \sigma, \quad y = u + \frac{1}{2}\sigma^2$

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(ii) $E(S_T) = E(S_0 e^{\mu t + \sigma B t})$

$= S_0 e^{\mu t} . E(e^{\sigma B t})$

$Bt \sim N(0,1)$

$E(e^{\sigma B t})$ is the MGF of Normal

$= e^{\mu + 1/2 \sigma^2 t^2}$

$= S_0 e^{\mu t} (e^{1/2 t \sigma^2})$

$= S_0 e^{t(\mu + 1/2 \sigma^2)}$

$Var(S_t) = E(S^2_T) - [E(S_T)]^2$

$= E(S^2_0 . e^{2\mu t + 2\sigma B t}) - [S_0 e^{t(\mu + 1/2 \sigma^2)}]^2$

$= S_0^2 e^{2\mu t} . E[e^{2\sigma B t} - (S_0 e^{t(\mu + 1/2 \sigma^2)})^2]$

$= S_0^2 e^{2\mu t} (e^{2\sigma^2} - e^{\sigma^2 t})$

(iii) $CoV(S_{t_1}, S_{t_2}) = E(S_{t_1}, S_{t_2}) - E(S_{t_1})(E(S_{t_2}))$

$E(S_{t_1}, S_{t_2}) = S_0 e^{\mu t_1} E(e^{\sigma B t_1}) S_0 e^{\mu t_2} E(e^{\sigma B t_2})$

$S_0^2 e^{\mu(t_1 + t_2)} . E(e^{\sigma B t} . t \sigma [Bt_1 + (Bt_2 - Bt_1)])$

The expectation sum is the MGF of normal distribution.

Therefore,

PTO

$= S_0^2\, e^{u(t_1+t_2)}\, e^{2\sigma^2 t_1} \cdot e^{1/2\sigma^2(t_2-t_1)}$

$= S_0^2\, e^{u(t_1+t_2)}\, e^{3/2\sigma^2 t_1 + 1/2\sigma^2 t_2}$

Q9

i) Forward price $= Se^{(rt)}$

S = Stock price, t = delivery, r = rf rate

Put call parity : $C + Ke^{-rt} = P + S$

C & P = Price of European call & put

K = Strike price, S = Stock price, t = Time to Expiry

$\therefore\ 13.334 + 70\,e^{(-0.25r)} = 0.120 + S$

$\therefore\ 8.869 + 75\,e^{(-0.25r)} = 0.568 + S$

By ~~solving~~ Solving the above eqⁿ.

$S = 82,\ r = 0.07$

$\therefore F = 82e^{(0.07 \times 0.25)}$

$\therefore\ F = 83.45$

(ii) $e^{(r_f \times 0.5)} = e^{(r_3 \cdot 0.25)} \cdot e^{(r_t \cdot 0.25)}$

$r_3 = 0.07$

by Substitution

$2.569 + 90\, e^{(-0.5 \times r_f)} = 7.909 + 82$

$r_f = 0.05$ & $r_t = 0.05$

(iii) By using Put call parity,

$1.899 + a e^{(-0.07 \times 0.25)} = 1.055 + 82$

$b + 80 e^{(-0.07 \times 0.25)} = 1.789 + 82$

$2.594 + 85 e^{(-0.07 \times 0.25)} = c + 82$

by solving the equations we get

$a = 77.5$

$b = 5.177$

$c = 4.119$

Q10 ∴ Rate of interest is 0, the risk neutral probability will be

$q = (1-d) / (u-d)$

for a recombining tree,

$d = 1/u$

$q = (1 - 1/u) / (u - 1/u)$

$= 1/u+1$

For no arbitrage opportunity we must have

$u > 1 > d$

Then $u > 1 \rightarrow u+1 > 2$

$q = \frac{1}{u+1} < \frac{1}{2}$

hence proved.

ii) from i, $q = 1/u+1$

$q = 1/2$

$\therefore u = 2$

$\therefore u = e^{(\sigma\sqrt{\Delta t})}$

$\therefore \sigma = 240.11\ \%$

Q 8 ~~sa~~ Current Stock price $= S_0$
option price $= t$
Duration $= T$

In our portfolio we will have short position in the option & long position in Δ shares.

If there is a upward movement the value of portfolio becomes $\Delta S_0 u - f_u$

If there is a downward movement the value of the portfolio becomes $\Delta S_0 d - f_d$

The two portfolios will be equal if,

$\Delta S_0 u - f_u = \Delta S_0 d - f_d$

or $\Delta = \frac{f_u - f_d}{S_0 - S_0 d}$

So, that portfolio is risk free therefore it must earn a risk free rate of interest.

$\therefore$ PV of Portfolio = $(\Delta S_0 u - f_u)e^{-rt}$

The cost of building the portfolio is $\Delta S_0 - f$

if the portfolio grows at rf rate,

$\Delta S_0 - f = (\Delta S_0 u - f_u)e^{-rt}$

$\therefore f = \Delta S_0 - (\Delta S_0 u - f_u)e^{-rt}$

$\therefore f = e^{-rt}[p f_u + (1-p) f_d]$

where $p = [e^{rt} - d]/[u-d]$

(ii) If there is an upward movement in the stock price the value of call option will increase & the value of put option decreases when the stock price goes down.
This happens because the value we are calculating the value of option in terms of the value of underlying asset.

(iii) $E(S_T)$ at time T
$= p S_0 u + (1-p) S_0 d \cdot 0.5$
or
$E(S_T) = p \cdot S_0(u-d) + S_0 d - 0.5$
Substitution p.
$E(S_T) = e^{rt} \cdot S_0$
By this we can interpret that the stock price grows at the risk free rate,

(iv) -

Q11

i) A recombining binomial tree is one which the sizes of ups & down steps is the same under all steps across all intervals.

Therefore the risk neutral probability (q') is also constant at all times & in all states. It has only (n+1) positive states of time as opposed to second possible states in a non recombining.

It assumes that the volatility and drift parameters of an underlying are constant over time.

(iv) (a) -

		1320
	1100	
		990
1000		
		1140
	950	
		855

$$q_1 = \frac{e^{(0.0175)} - 0.95}{1.10 - 0.95} = 0.4510$$

$$q_2 = \frac{e^{0.025} - 0.90}{1.20 - 0.90}$$

$$= 0.4172$$

put payoffs at the states are 0, 0, 0, 95.

The value of option $V_1(1)$ following

an up step over the first 3 months is

$V_1 e^{0.025} = [0.41772 \times 0] + [0.058228 \times 0]$

$\therefore V_1(1) = 0$

The value of option following a down step over the first three months is

$V_2(2) e^{0.025} = [0.41772 \times 0] + [0.58228 \times 95]$

$= 53.9502$

The current value is

$V_0 e^{(0.0175)} = [0.4510 \times 0] + [0.5490 \times 53.9508]$

$= 29.105$

Q12) i) $z(t)$ is a standard Brownian

(a) $du(t) = 2dz_t - 0$
$= 0dt + 2dz_t$

Thus $(u(t))$ has zero drift

(b) $dv(t) = d[z(t)]^2 - dt$
$d[z(t)]^2 = 2z(t) \cdot dz_t + dt$

Thus, $dv(t) = 2z_t \cdot dz_t$ Thus stochastic process $v(t)$ has 0 drift.

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(c) $dw(t) = d[t^2 z(t)] - 2t z(t)\, dt$

$\therefore d[t^2 z(t)] = t^2 dz(t) + 2t z(t)\, dt$

$dw(t) = t^2 dz(t)$

$\therefore$ The process $w(t)$ has 0 drift

Q13) Let $\frac{S_t}{S_0} \sim LN\left[\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right]$

Expected return $= \mu$
Volatility $= \sigma$

Expected value at first time step on the tree $= qS_0 u + (1-q) S_0 d$

$S_0 e^{\mu dt} = qS_0 u + (1-q) S_0 d$

$q = (e^{\mu dt} - d) / (u - d)$

standard deviation of stock price $= \sigma\sqrt{\delta t}$

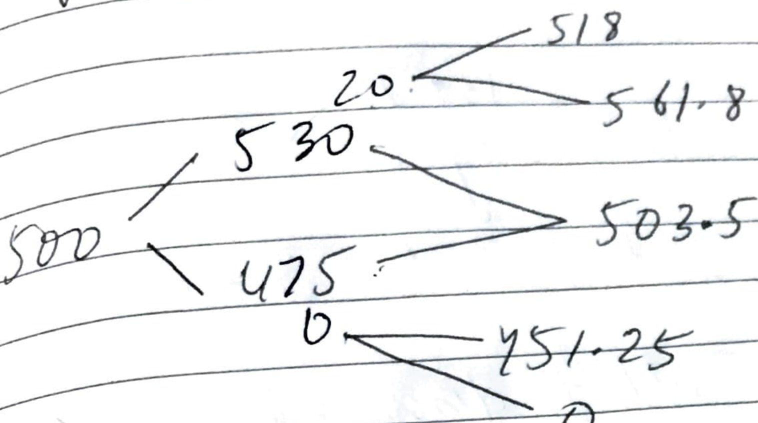
The variance is,

$q u^2 + (1-q) d^2 - [qu + (1-q)d]^2 = \sigma^2 dt$

$\therefore e^{\mu dt}(u + d) - ud - e^{2\mu dt} = \sigma^2 dt$

$u = e^{\sigma\sqrt{dt}}$ and $d = e^{-\sigma\sqrt{dt}}$

1.4 (i) Pay off Diagram for European Call



$u = 1.06, \quad d = 0.95, \quad r = 0.05, \quad t = 0.25$

$p = e^{(rt)} - d \,/\, (u - d)$

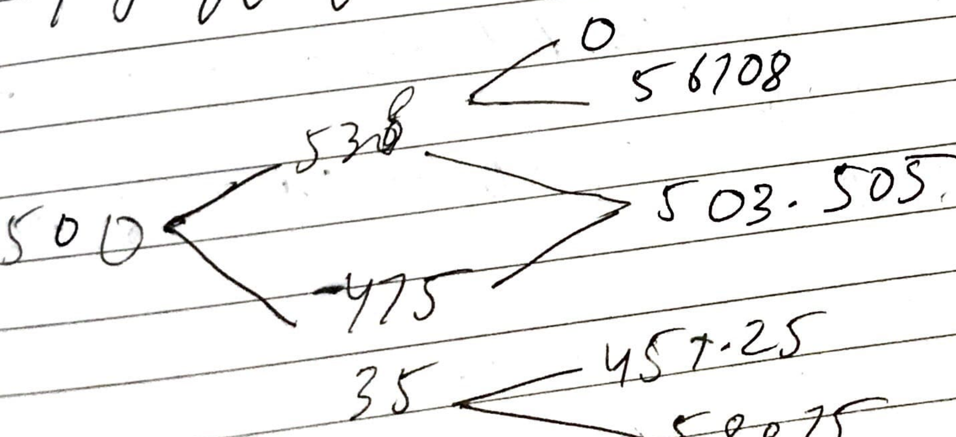
$= 0.\cancel{0}5689$

$1 - p = 0.4311$

Value of European Call $e^{(-0.5 \times 0.05)} \times 51.8 \times 0.5689$

$= 16.351$

(ii) Pay off for put



Value of European put $= e^{(-0.05 \times 0.5)} \times 0.4311^2 \times 58.75 \times 2 \times 0.4311 \times 0.5689 \times 6.50$

$= 13.759$

Using Put call parity,

Value of put + Stock = 513.759

Value of call + discounted price = 513.7

(iii) For American option the value at the final node is same for a European option.

Value at Node B $= 6.50 \times e^{(-0.25 \times 0.05)} \times 0.4311$
$= 2.767$

Value at Node C $= e^{(-0.25 \times 0.05)} (6.50 \times 0.5689 + 58.75 \times 0.4311)$
$= 28.66442$

Value of American put option at the t_0

$= e^{(-0.05 \times 0.25)} (35 \times 0.4311 + 2.767 + 0.5869)$
$= 16.46$

(IV) real rate of return = 0.09

$p = 0.6614$

Expected pay off $= 20 \times 0.6619$
$= 13.228$

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Q15

(1) $f = f(S_t, t) = S^k$

$$\frac{df}{dS} = KS^{K-1}, \quad \frac{d^2f}{dS^2} = K(K-1)S^{K-2}$$

$$\frac{df}{dt} = 0$$

$$df = KS^{K-1}dS_t + \frac{1}{2}K(K-1)S^{K-2}(dS_t)^2$$

$$= f\left[K\mu + \frac{1}{2}K(K-1)\sigma^2\right]dt + f[K\sigma]dW_t$$

Hence

$f = S^K$ is GBM

with drift $= K\mu + \frac{1}{2}K(K-1)\sigma^2$

Volatility $= K\sigma$

$$f_t = f_0 e^{(\mu - \frac{1}{2}\sigma'^2)t + \sigma' W_t}$$

This means $\frac{f}{f_0}$ is $LN\left[\left(\mu' - \frac{1}{2}\sigma'^2\right)t, \sigma'^2_t\right]$

$$E(f) = f_0 e^{\mu' t}, \quad V(f) = f_0 e^{2\mu' t}(e^{\sigma'^2 t} - 1)$$

(ii) Let $f = f(t, S_t) = e^{-rt} S_t$.

$$\frac{df}{dt} = -r e^{-rt} S_t, \quad \frac{df}{dS} = e^{-rt}, \quad \frac{d^2 f}{dS^2} = 0$$

By Ito's Calculus

$$df = -r e^{-rt} S_t \, dt + e^{-rt} dS_t$$

$$= (\mu - r) f \, dt + \sigma f \, dW_t$$

Which is a martingal if $\mu = r$.

(iii) S^k will be a martingal is ~~[illegible]~~

$$k\mu + \frac{1}{2} k(k-1)\sigma^2 = r$$