Nimitt Chauhan
Roll no. 11

PAGE No. / DATE / /

Probability & Statistics
Assignment-1

1.) $x \sim Poi(\theta)$, where θ is large,

$\therefore Z = \dfrac{\hat{\lambda} - \lambda}{\sqrt{\hat{\lambda}/n}} \sim N(0,1)$ approx.

$\hat{\lambda} = \dfrac{\sum x_i}{n}$, $\hat{\lambda} = 200$

$$P\left(Z_{2.5\%} < Z < Z_{97.5\%}\right) = 0.95$$

$$P\left(-1.96 < \frac{\hat{\lambda} - \lambda}{\sqrt{\hat{\lambda}/n}} < 1.96\right) = 0.95$$

$$P\left(-1.96 < \frac{200 - \lambda}{\sqrt{200}} < 1.96\right) = 0.95$$

$$P\left(-1.96\sqrt{200} - 200 < -\lambda < 1.96\sqrt{200} - 200\right) = 0.95$$

$$P\left(200 - 1.96\sqrt{200} < \lambda < 200 + 1.96\sqrt{200}\right) = 0.95$$

$\therefore$ 95% CI for $\lambda = \left(200 - 1.96\sqrt{200},\ 200 + 1.96\sqrt{200}\right)$

$= (172.2814, 227.7186)$

2.) $X_i \ni i = 1, 2, \ldots n$

mean $= \mu$, var $= \sigma^2$

we know - $\bar{X} = \frac{\Sigma x_i}{n}$, $s^2 = \frac{1}{n-1} (\Sigma x_i^2 - n\bar{x}^2)$

(I) $E(\bar{X}) = E\left(\frac{\Sigma x_i}{n}\right)$

$E(\bar{x}) = \frac{E(\Sigma X_i)}{n} = \frac{n\mu}{n}$

$\therefore \boxed{E(\bar{X}) = \mu}$

$V(\bar{X}) = V\left(\frac{\Sigma x_i}{n}\right)$

$V(\bar{X}) = \frac{V(\Sigma x_i)}{n^2} = \frac{\Sigma V(x_i)}{n^2}$

$V(\bar{X}) = \frac{n\sigma^2}{n^2}$

$\therefore \boxed{V(\bar{X}) = \frac{\sigma^2}{n}}$

(II) $E(s^2) = E\left[\frac{1}{n-1}(\Sigma x_i^2 - n\bar{x}^2)\right]$

$E(s^2) = \frac{1}{n-1} E\left[(\Sigma x_i^2 - n\bar{x}^2)\right]$

$E(s^2) = \frac{\Sigma E(x_i^2) - nE(\bar{x}^2)}{n-1}$

we know-

$$V(x) = E(x_i^2) - E(x)^2$$

$$\therefore \sigma^2 = E(x_i^2) - \mu^2$$

$$\therefore E(x_i^2) = \sigma^2 + \mu^2$$

similarly, $E(\bar{x}^2) = \mu^2 + \dfrac{\sigma^2}{n}$

$$\therefore E(s^2) = \frac{1}{n-1}\left[\Sigma(\mu^2+\sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right]$$

$$E(s^2) = \frac{1}{n-1}\left[n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2\right]$$

$$E(s^2) = \frac{1}{n-1}(n-1)\sigma^2$$

$$\boxed{E(s^2) = \sigma^2}$$

Hence proved.

(iii) $X \sim N(\mu, \sigma^2)$

$\therefore \dfrac{(n-1)s^2}{\sigma^2} = \chi^2_{n-1}$, To find $V(s^2) = \dfrac{2\sigma^4}{n-1}$

$$\therefore V\left(\frac{(n-1)s^2}{\sigma^2}\right) = \frac{(n-1)^2}{\sigma^4} \cdot V(s^2)$$

But we know, $\dfrac{(n-1)s^2}{\sigma^2} = \chi^2_{n-1}$

$$\therefore V(\chi^2_{n-1}) = \frac{(n-1)^2 \cdot V(s^2)}{\sigma^4}$$

$$\therefore 2(n-1) = \frac{(n-1)^2 \cdot V(s^2)}{\sigma^4}$$

$$\therefore \boxed{V(s^2) = \frac{2\sigma^4}{(n-1)^2}}$$

(IV) $n = 10$, $\sigma^2 = 100$

$$P(\sigma^2 - 50\%\cdot\sigma^2 < s^2 < 50\%\cdot\sigma^2 + \sigma^2)$$

$$\therefore P(0.5\sigma^2 < s^2 < 1.5\sigma^2)$$

$$= P(s^2 < 1.5\sigma^2) - P(s^2 < 0.5\sigma^2)$$

$$= P(s^2 < 150) - P(s^2 < 50)$$

$$= P\left(\frac{9s^2}{\sigma^2} < \frac{150\times 9}{100}\right) - P\left(\frac{9s^2}{\sigma^2} < \frac{50\times 9}{100}\right)$$

$$= P\left(\chi^2_9 < 13.5\right) - P\left(\chi^2_9 < 4.5\right)$$

$$= 0.8587 - 0.1245$$

$$\therefore \boxed{P(50 < s^2 < 150) = 0.7342}$$

3.) $X \sim Bin(4, p)$

(1) $$L = \prod_{i=1}^{180} f(x_i)$$

$$L = P(X=0)^{86} \times P(X=1)^{75} \times P(X=2)^{16} \times P(X=3)^2 \times P(X=4)^1$$

$$L = \left[{}^4C_0\, p^0 (1-p)^4\right]^{86} \times \left[{}^4C_1\, p^1 (1-p)^3\right]^{75} \times \left[{}^4C_2\, p^2 (1-p)^2\right]^{16} \times \left[{}^4C_3\, p^3 (1-p)\right]^2 \times \left[{}^4C_4\, p^4 (1-p)^0\right]^1$$

$$L = C \times p^{117} \times (1-p)^{603}$$

where C is constant.

$$\log L = \log(C) + 117\log(p) + 603\log(1-p)$$

PAGE No.
DATE / /

differentiating w.r.t p -

$$\frac{dL}{dp} = 0 + \frac{117}{p} - \frac{603}{(1-p)}$$

$$\therefore \frac{117}{p} - \frac{603}{(1-p)} = 0$$

$$117 - 117p = 603p$$

$$\therefore p = \frac{117}{720}$$

$$\therefore \boxed{p = 0.1625}$$

(11) H_0 = no. of claims coform to binomial
H_1 = no. of claims not coform binomial.

	0	1	2	3	4
O_i	86	75	16	2	1
E_i	88.542	68.724	19.998	2.574	0.108

$$P(X=x) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

$\therefore P(X=0) = 0.4919$
$P(X=1) = 0.3818$
$P(X=2) = 0.1111$
$P(X=3) = 0.0143$
$P(X=4) = 0.0006$

O_i	86	75	19
E_i	88.542	68.724	22.68

$$T.S. = \frac{\Sigma(O_i - E_i)^2}{E_i} \sim \chi^2_1$$

$$T.S. = 1.243223$$

$\therefore$ we can accept that it follows binomial model.

4) $$f(x) = \frac{c\beta^3}{(x+\beta)^4}, \quad x>0, \beta>0$$

(i) Given P.D.F looks like pareto distribuition with PDF -

$$\cancel{f(x)} \quad PDF = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}$$

$\therefore \beta = \lambda$ here

$\therefore c = \alpha$ here

$\therefore$ ~~c = 3~~ $\boxed{c=3}$

$$E(x) = \frac{\lambda}{\alpha-1} = \frac{\beta}{2}$$

$$V(x) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = \frac{3\beta^2}{4}$$

(ii) $$E(x) = \frac{\beta}{2}$$

$$\therefore \bar{x} = \frac{\beta}{2}$$

$$\therefore \hat{\beta} = 2\bar{x}$$

Hence proved.

Bias:

$$Bias = E(\hat{\beta}) - \beta$$
$$Bias = E(2\bar{x}) - \beta = 2\, E(\bar{x}) - \beta$$
$$Bias = \frac{2}{n} E(x) - \beta = \frac{2}{n} \times \frac{n\beta}{2} - \beta$$
$$\boxed{Bias = 0}$$

Consistency -

$$MSE = V(\hat{\beta}) + Bias^2$$
$$MSE = V(\hat{\beta})$$
$$MSE = V(2\bar{x})$$
$$MSE = 4\, V(\bar{x})$$
$$MSE = \frac{4}{n^2} \Sigma V(x_i) = \frac{4}{n^2} \frac{3\beta^2 n}{4}$$
$$MSE = \frac{3\beta^2}{n}$$

$\therefore$ as n increases, MSE closes to 0

$\therefore$ $\hat{\beta}$ is consistent.

(iii) $\hat{\beta} = b\bar{x}$

$$E(\hat{\beta}) = E(b\bar{x})$$
$$= bE\left(\frac{\Sigma x_i}{n}\right) = \frac{b}{n} \Sigma E(x_i)$$
$$E(\hat{\beta}) = \frac{b\beta}{2}$$

$$V(\hat{\beta}) = V(b\bar{x})$$
$$= \frac{b^2 \Sigma V(x_i)}{n^2}$$

$$= \frac{3}{4}\frac{b^2\beta^2}{n}$$

$$MSE = V(\beta) + Bias^2$$

$$= \frac{3b^2\beta^2}{4n} + \left(\frac{b\beta - 2\beta}{2}\right)^2$$

$$= \frac{3b^2\beta^2}{4n} + \frac{b^2\beta^2 + 4\beta^2 - 4\beta^2 b}{4}$$

$$MSE = \frac{\beta^2}{4n}\left(3b^2 + nb^2 + 4n - 4nb\right)$$

diff. wrt b -

$$\frac{dm}{db} = \frac{\beta^2}{4n}\left(6b + 2nb - 4n\right)$$

$$\frac{\beta^2}{4n}\left(6b + 2nb - 4n\right) = 0$$

$$\therefore \quad 6b + 2nb - 4n = 0$$

$$b(3+n) = 2n$$

$$\therefore b = \frac{2n}{3+n} = \frac{2}{\left(1+\frac{3}{n}\right)}$$

$$\frac{d^2m}{db^2} = \frac{\beta^2}{4n}\left(6+2n\right) > 0$$

$$\therefore \quad b = \frac{2}{\left(1+\frac{3}{n}\right)} \quad \text{minimizes MSE}$$

PAGE No.
DATE / /

5.)

(i) $X \sim U(a, b)$

$$E(X) = \frac{a+b}{2} = \bar{x}$$

$$\therefore b = 2\bar{x} - a \quad \text{--- (1)}$$

$$V(X) = \frac{(b-a)^2}{12} = S^2$$

$$\therefore 12S^2 = (b-a)^2$$

$$\therefore 12\sqrt{3}\,S = b - a$$

$$12\sqrt{3} \cdot S = 2\bar{x} - 2a$$

$$\therefore \sqrt{3}\,S = \bar{x} - a$$

$$\therefore \boxed{\hat{a} = \bar{x} - \sqrt{3}S}$$

(ii) $\hat{b} = 2\bar{x} - a$

$$\therefore \boxed{\hat{b} = \bar{x} + \sqrt{3}S}$$

(iii) sample observations $- \{10, 20, 30, 40, 50\}$

$\bar{x} = 30$, $S =$

PAGE No.
DATE / /

6.) $H_0 = p = 0.1$
$H_1 = p > 0.1$

(I) $P(\text{type I error}) = P(\text{rejecting } H_0 / H_0 \text{ is true})$
$= P(X \geq 3 \mid p=0.1) + P(X=0 \mid p=0.1) \times P(Y \geq 5 \mid p=0.1) + P(X=1 \mid p=0.1) P(Y \geq 4 \mid p=0.1) + P(X=2 \mid p=0.1) \cdot P(Y \geq 3 \mid p=0.1)$

$= (1 - 0.8891) + (0.2824 \times 0.0043) + (0.3766 \times 0.2566) + (0.2301 \times 0.1109)$
$= 0.147268 \approx 15\%$

(II) ~~P(rejecting~~
$P(\text{rejecting when } p=0.3) = P(X \geq 3 \mid p=0.3) + P(X=0 \mid p=0.3) P(Y \geq 5 \mid p=0.3) + P(X=1 \mid p=0.3) P(Y \geq 4 \mid p=0.3) + P(X=2 \mid p=0.3) P(Y \geq 3 \mid p=0.3)$
$= 0.912532$

(III) $P(\text{type II error}) = 1 - 0.912532$
$= 0.87468$

PAGE No.
DATE / /

(iv) $\dfrac{X/n - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$

$n = 120$, $X = H_o$

$\hat{p} = \dfrac{40}{120} = \dfrac{1}{3} = 0.3333$

$\therefore Z_{10\%} = 1.2816$

$\hat{p} - 1.2816\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} = 0.3333 - 1.2816\sqrt{\dfrac{0.3333(0.6667)}{120}}$

$= \boxed{0.2782}$

(v) $p > 0.2782$ at 10% L.O.S.

$H_o \Rightarrow p = 0.2782$, $H_1 \Rightarrow p > 0.2782$

$\dfrac{X - np_o}{\sqrt{np_oq_o}} = \dfrac{39.5 - 120(0.2782)}{\sqrt{120 \times 0.2782 \times 0.7218}}$

$= \boxed{1.2511}$

$\therefore$ we can not reject H_o

(vii)

	I_1	I_2
n	12	15
$\bar{X}$	65	70
S_1	54	70

$S_p^2 = \dfrac{1}{25}\left(11(2916) + 14(4800)\right) = 4027.04$

$\dfrac{65 - 70}{65.459\sqrt{\frac{9}{60}}} = -0.2034$

$\therefore \pm 2.060$ at 5% level,

7.) (I)

	1 Public	2 Private
$\bar{X}$	30	35.0555
S^2	64.3125	67.4027

(II) if populations have equal variance, t-test can be used.

(III) $H_0 \Rightarrow \mu_1 = \mu_2$

$H_1 \Rightarrow \mu_2 > \mu_1$

$$TS = \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{Sp\sqrt{\frac{n_1 + n_2}{n_1 n_2}}}$$

$$TS = \frac{35.0555 - 30}{8.1152\sqrt{\frac{2}{9}}}$$

$$\boxed{TS = 1.32151}$$

tabulated value $t_{16, 5\%} = 1.746$

$\therefore$ TS < tabulated value

$\therefore$ H_0 isnt rejected.

PAGE No.	
DATE	/ /

8.) (i) unbiased estimator : it is an estimator whose expected value is equal to parameter. It is possible when bias is equal to 0

(ii) bias : it is the positive or negative difference in the estimated and actual value.

(iii) $MSE(\hat{\theta}) = var(\theta) + bias^2$

(iv) $\tilde{\theta}$ is more efficient as MSE value is lower compared to $\hat{\theta}$.

(v) ~~Both~~ Estimators will be called consistent when their value is close to 0.
In this case $\tilde{\theta}$ is more consistent than $\hat{\theta}$.

(vi) Maximum likelihood estimate and method of moments.

9.) (i) $t_k \Rightarrow \frac{N(0,1)}{\sqrt{\chi^2_k/k}}$ where k is degree of freedom.

(ii) $k > 2$

$E(t_k) = 0$, $V(t_k) = \frac{k}{k-2}$

(iii) [a] $\frac{\bar{x} - \mu}{S/\sqrt{n}} \sim t_{n-1}$, $n = 10$

$$P\left(t_{9,0.5\%} < \frac{\bar{x} - \mu}{S/\sqrt{n}} < t_{9,99.5\%}\right) = 0.99$$

DATE / /

$$P\left(\bar{X} - t_{9, 0.5\%} \cdot \frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{9, 0.5\%} \cdot \frac{S}{\sqrt{n}}\right) \geq 0.99$$

$$\therefore t_{9, 0.5\%} = 3.250$$

$\therefore$ 99% confidence interval -

$$\mu = \left(50 - \frac{3.25\sqrt{48.667}}{\sqrt{10}}, 50 + \frac{3.25\sqrt{48.667}}{\sqrt{10}}\right)$$

$$\mu = (42.83, 57.17)$$

[b] $t_{n-1} \sim N\left(0, \sqrt{\frac{n-1}{n-3}}\right)$

$$t_{n-1} \sim N\left(0, \sqrt{\frac{9}{7}}\right)$$

$$\therefore \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N\left(0, \sqrt{\frac{9}{7}}\right)$$

$$\therefore \frac{\bar{X} - \mu}{S/\sqrt{7n/9}} \sim N(0, 1)$$

critical value : 2.58

$$\text{confidence interval} = \left(50 - 2.58\sqrt{\frac{48.667 \times 9}{10 \times 7}}, 50 + 2.58\sqrt{\frac{48.667 \times 9}{70}}\right)$$

$$= (43.546, 56.4535)$$

PAGE No.
DATE / /

10) $n = 1000$

$\lambda = 3$

$x \sim Poi(n\lambda)$; $Poi(3000)$

(I) $P(\text{type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true})$

$= P(x > 3100 \mid X \sim Poi(3000))$

$\therefore P(x > 3100) = P\left(Z > \frac{3100 - 3000}{\sqrt{3000}}\right)$

$= 1 - P(Z < 1.825)$

$P(x > 3100) = 0.0339$

(II) Type II error is when hypothesis is false.

(III) Power of list $= P(\text{rejecting } H_0 \mid H_0 \text{ is false})$

$= P(x > 3100 \mid X \sim Poi(n\lambda) - N(n\lambda, n\lambda))$

$= 1 - P\left(Z < \frac{3100 - n\lambda}{\sqrt{n\lambda}}\right)$

(IV) $\hat{\lambda} =$

$P\left(-2.5758 < \frac{2.9 - \mu}{\sqrt{2.9/1000}} < 2.5758\right) = 0.99$

$\therefore$ confidence interval for

$\mu = (2.7613, 3.0387)$

PAGE No.

DATE / /

11.) $n = 12$, $\Sigma x = 213200$, $s = 10,144$.
$\Sigma x^2 = 4919860000$, $\bar{x} = 17,767$

$$\Sigma x_1 = \Sigma x - 11000 - 48000 + 27500 + 31500$$
$$= 213200$$

$$\text{new } \bar{x}' = \frac{213200}{12}$$

$$\boxed{\text{new } \bar{x} = 17,767}$$

$$\Sigma x_1^2 = 49198600 - (11000)^2 - (48000)^2 + (27500)^2 + (31500)^2$$
$$\Sigma x_1^2 = 4243360000$$

$$\text{new } s^2 = \frac{1}{11}\left(4243360000 - 12(17767)^2\right)$$
$$= 41396775.64$$

$$\therefore \boxed{\text{new } s = 6434.032611}$$