Financial Engineering – 1 Assignment 2;

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Q.1

(i)

S0 = £65

σ =25% p.a.

r = 2% p.a.

X= £55

T= 6 months

$$c_t = S_0 * \emptyset(d_1) - X * e^{-rt} * \emptyset(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$$d_2 = d_1 - \sigma * \sqrt{T}$$

$$c_0 = £11.41871$$

(ii)

The delta of a call option is defined as the change in the price of the call option with respect to the change in the price of the underlying.

$$\Delta_c = \frac{dc_t}{dS_t}$$

(iii)

$$\emptyset(d_1) = 0.862134$$

(iv)

Using put-call parity,

$$\Delta_p + 1 = \Delta_c$$

$$\Delta_p = -0.1379$$

Q.2

(i)

The delta of an option is defined as the change in the price of the option with respect to the change in the price of the underlying.

$$\Delta_c = \frac{dc_t}{dS_t}$$

Vega of an option is defined as the change in the price of the option with respect to the change in the volatility of the underlying.

$$V_c = \frac{dc_t}{d\sigma}$$

(ii)

Using the put-call parity,

$$p_t + S_0 = c_t + K * e^{-rt}$$

Differentiating w.r.t. σ

$$V_p = V_c$$

Hence Proved.

(iii)

S0 = $55

X = $50

σ = 25%

r = 5%

T = 1 year

$$c_t = S_0 * \text{Ø}(d_1) - X * e^{-rt} * \text{Ø}(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$$d_2 = d_1 - \sigma * \sqrt{T}$$

$$d_1 = 0.7062 \text{ and } d_2 = 0.4562$$

$$\text{Thus}, c_t = \$9.6526$$

Using the put-call parity,

$$p_t = \$2.214017$$

(iv)

For a portfolio to be 'delta-hedged', it means that for a change in the price of the underling, the value of the derivative does not change i.e., it is not sensitive to the change in the price of the underlying.

Similarly, for a portfolio to be ‘Vega-hedged’, it means that for a change in the volatility of the underlying, the value of the derivative does not change i.e., it is not sensitive to the change in the volatility of the underlying.

Q.3

(i)

The price of the derivative at time t is given by:

$$\text{Price} = e^{-r(T-t)} * E_Q[X_T|F_t]$$

(ii)

S0 = £50

X = £49

r = 5% p.a.

σ = 25% p.a.

T = 6 months

Value of the European call option, assuming Black Scholes model holds true,

$$c_t = S_0 * \emptyset(d_1) - X * e^{-rt} * \emptyset(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$$d_2 = d_1 - \sigma * \sqrt{T}$$

$$d_1 = 0.3441 \text{ and } d_2 = 0.1673$$

$$\text{Thus}, c_t = £4.6604$$

(iii) Value of the American call option = Value of the European call option = £4.6604

(iv) Using the put-call parity,

$$p_t + S_0 = c_t + K * e^{-rt}$$

$$p_t = £2.4506$$

(v) If dividends were payable, then this would cause the value of the underlying asset to fall, each time by the amount of dividend payable.

The value of the European call option would decrease, as having the option to buy a share which would be less, for a fixed price at the expiry date, would be less valuable.

The value of the American call would increase relative to the European call.

Q.4

(i)

The CMG theorem states that: Suppose Z_t is a SBM under P. And there exists a measure Q such that P and Q are equivalent measures then,

$$\mathrm{Zbar}(t) = Z_t + \gamma t$$

(ii)

The discounted value of a security price process is a martingale under the risk neutral measure.

Q.5

(i)

The delta of an option is defined as the change in the price of the option with respect to the change in the price of the underlying.

$$\Delta_c = \frac{dc_t}{dS_t} = \emptyset(d_1)$$

(ii)

S0 = $40

r = 2% p.a.

X = $45.91

T = 5 years

Δ = 0.6179

$$c_t = S_0 * \emptyset(d_1) - X * e^{-rt} * \emptyset(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$$d_2 = d_1 - \sigma * \sqrt{T}$$

$$\emptyset(d_1) = \emptyset(0.3) = 0.6179$$

$$d_1 = 0.3 \text{ – From tables}$$

$$0.3 = \frac{\ln\left(\frac{40}{45.91}\right) + \left(2\% + \frac{1}{2} * \sigma^2\right) * 5}{\sigma * \sqrt{5}}$$

$$\sigma = 32\%$$

(iii)

The general risk-neutral pricing formula for a derivative that pays an amount X_T at time T is given by:

$$V_0 = e^{(-rT)} * E_Q[X_T|F_0]$$

Since, the stock prices are independent,

$$V_0 = e^{-rT} * c * Q\left[\frac{S_1}{S_0} < k_S\right] * Q\left[\frac{R_1}{R_0} < k_R\right]$$

(iv)

For two perfectly correlated stock prices S_t and R_t, then

$$\frac{S_1}{S_0} = \frac{R_1}{R_0}$$

Thus, the equation for V_0 can be written as:

$$V_0 = e^{-rT} * c * Q\left[\frac{S_1}{S_0} < k_S, \frac{S_1}{S_0} < k_R\right]$$

$$V_0 = e^{-rT} * c * Q\left[\frac{S_1}{S_0} < \min(k_S, k_R)\right]$$

$$V_0 = e^{-rT} * c * Q\left[\frac{R_1}{R_0} < \min(k_S, k_R)\right]$$

(v)

We know, under the Black Scholes option pricing model,

$$1 - \emptyset(d_2) = \emptyset(-d_2) = \emptyset\left(\frac{\ln\left(\frac{X}{S_0}\right) - \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}\right)$$

$$V_0 = e^{-rT} * c * Q[S_1 < S_0 k_S] * Q[R_1 < R_0 k_R]$$

$$V_0 = \$1.6075$$

Q.6

(i) (a)

Delta for a put option is given as:

$$\Delta_p = \emptyset(d_1) - 1$$

(b)

Since, it's a delta-hedged portfolio,

$$V_0 = \psi - 24830 * S_0$$

$$\Delta_V = -24830$$

For the delta of the portfolio to replicate the delta of the put option we set $100000\Delta_p = \Delta_V$

$$\Delta_P = \frac{\Delta_V}{100000} = -0.2483$$

(ii)

Since,

$\Delta_p = \emptyset(d_1) - 1 = -0.2483$

$\emptyset(d_1) = 0.7517$

$d_1 = 0.68$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

Solving the above equation by substituting the values, we get

$\sigma = 7.1\%$

(iii) (a)

$p_t = K * e^{-rT}\, \emptyset(-d_2) - S_0 * \emptyset(-d_1)$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$d_2 = d_1 - \sigma * \sqrt{T}$

$p_t = £0.0696$

(iii) (b)

$V_0 = \psi - 24830 * 6.40$

$\psi = £165{,}872$

Q.7

(i) The delta of a derivative is defined as the change in the price of the derivative with respect to the change in the price of the underlying.

Gamma of a derivative is defined as the change in the delta of the derivative with respect to the change in the price of the underlying.

Vega of a derivative is defined as the change in the price of the underlying with respect to the change in the volatility of the underlying.

(ii) Given data:

$\text{Since, delat for a call option under the Black Scholes option pricing model} = \emptyset(d_1) = 0.80106.$

(iii) The replicating portfolio is constructed using let say ψ amount of the cash and ϕ units of the underlying.

$V_0 = \psi + \phi * S_0$

The delta for this portfolio is given as,

$\Delta_v = \phi = 0.801$

Vega of the replicated portfolio is equated to 0.801 because of the reason it being a delta-hedged portfolio.

$V_0 = \psi + 0.801 * S_0$

$V_0 = \psi + 48.06$

$\text{Thus}, 17.91 = \psi + 48.06$

$\psi = -30.15$

Thus, the portfolio contains 0.801 units of the share and a short position in cash of amount $30.15

(iv)

We know,

$$\frac{\Delta c}{\Delta \sigma} = \frac{dc}{d\sigma}$$

$$\frac{c_2 - c_1}{2\%} = 29$$

$c_2 = \$18.49$

Q.8

(i)

The delta of an option is defined as the change in the price of the option with respect to the change in the price of the underlying.

$$\Delta_c = \frac{dc_t}{dS_t} = \emptyset(d_1)$$

(ii)

Given data:

$S_0 = \$100$

$r = 3\%$

$X = \$109.42$

$T = 1 \text{ year}$

$\Delta_c = 0.42074$

$\phi(-0.20) = 0.42074$

$\text{Thus}, d_1 = -0.20$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$\sigma = 20\%$

Q.9

Answer: (i)

The function g must satisfy:

$$\frac{dg}{dt} + (r-q) * S_t \frac{dg}{dS_t} + \frac{1}{2} * \sigma^2 * S_t^2 * \frac{d^2g}{dS_t^2} = rg$$

The boundary condition applies at maturity and is $D_T = g(T, S_t) = f(S_T)$

(ii)

Suppose, $D_t = g(t, S_t) = \frac{S_t^n}{S_0^{n-1}} e^{\mu(T-t)}$ with n>1

Then $D_T = g(T, S_T) = f(S_T) = \frac{S_T^n}{S_0^{n-1}}$, so the boundary condition is satisfied.

The partial derivatives in the PDE in question (i) are given by,

$$\frac{dg}{dt} = -\mu g$$

$$\frac{dg}{dS_t} = \frac{n}{S_t} g$$

$$\frac{d^2g}{dS_t^2} = \frac{n(n-1)}{S_t^2} g$$

Substituting,

$$-\mu g + (r-q) * S_t * \frac{n}{S_t} g + \frac{1}{2} \frac{\sigma^2 S_t^2 n(n-1)}{S_t^2} g = rg$$

$$\mu = (r-q)n - r + \frac{1}{2}\sigma^2 n(n-1)$$

Q.10

→

(i)

Consider a portfolio which is long one call and cash of $K * e^{r(T-t)}$ and short one put

The portfolio has a payoff at the time of expiry of S_T

$c_t + Ke^{r(T-t)} - P_t = S_t$

(ii)

Given data:

$X = \$120$

$T = 1$ year

$c_t = \$10.09$

$r = 2\%$ p.a.

$S_0 = \$110$

$c_t = S_0 * \emptyset(d_1) - X * e^{-rt} * \emptyset(d_2)$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$d_2 = d_1 - \sigma * \sqrt{T}$

$\sigma = 30\%$

(iv) (a)

The payoff from the portfolio D, satisfy,

$S_1 - 121 \leq D \leq S_1 - 120$

It follows that the initial price, V, of the portfolio should satisfy,

$S_0 - 121e^{-r} \leq V \leq S_0 - 120e^{-r}$

$\text{i.e.}, -8.604 \leq V \leq -7.624$

(b)

And this implies that $17.714 \leq P_0 \leq 18.6914$

(v)

The Black-Scholes price (using the formula) is \$18.35

Q.11

(i)

Given data:

$X = \$150$

$r = 2\%\ \text{p.a.}$

$S_0 = \$117.98$

We know, $\Delta_{portfolio} = 100000\Delta_c - 18673 * \Delta_s$

But, $\Delta_s = 1$ and $\Delta_{portfolio} = 0$

Thus, $\Delta_c = 0.18673$

(ii)

Since, under the Black Scholes option pricing model, $\Delta_c = \phi(d_1)$

$\phi(d_1) = 0.18673$

Thus, $d_1 = -0.89$

Using the black sholes option pricing formula,

$$c_t = S_0 * \emptyset(d_1) - X * e^{-rt} * \emptyset(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$$d_2 = d_1 - \sigma * \sqrt{T}$$

$$\sigma = 22\% \text{ p.a.}$$

(iii)

Using the black sholes option pricing formula,

$$p_t = K * e^{-rT} \emptyset(-d_2) - S_0 * \emptyset(-d_1)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$$d_2 = d_1 - \sigma * \sqrt{T}$$

$$p_t = \$31.45$$

(iv)

Taking partial derivatives of the put-call parity relationship with respect to S_0 gives,

$$\Delta_c = \Delta_p + \Delta_s$$

$$\gamma_c = \gamma_p + \gamma_s$$

So, the investor must have a short position in 100,000 put options.

If we let x be the number of units of stock held by the investor, the total delta for the portfolio is given by,

$$\Delta_{portfolio} = 100000\Delta_c - 100000\Delta_p + x\Delta_s = 0$$

$$x = -100000$$

Q.12

(i)

The main assumptions underpinning the Black-Scholes model are as follows:

- No taxes or transaction costs.
- Complete divisibility of holdings is allowed.
- Unlimited buying and selling.
- Underlying asset follows a continuous path.
- Geometric Brownian motion.
- The risk-free rate and the volatility of the underlying asset is constant.
- Investors are rational and risk-averse.
- $dS_t = \mu S_t dt + \sigma S_t dB_t$ where B_t is a SBM

(ii)

$S_0 = £8$

$X = £9$

$r = 2\% \text{ p.a.}$

$\sigma = 20\% \text{ p.a.}$

$T = 3 \text{ months} = \frac{3}{12} \text{years}$

Using the black sholes option pricing formula,

$p_t = K * e^{-rT}\, \emptyset(-d_2) - S_0 * \emptyset(-d_1)$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2} * \sigma^2\right) * T}{\sigma * \sqrt{T}}$$

$d_2 = d_1 - \sigma * \sqrt{T}$

$p_t = £1.01$

(iii) The risk-free rate and the put option price are inversely related.

--- Thank you ---