ASSIGNMENT 2
PROBABILITY AND STATISTICS

1. Let E be the claim size of home insurance
Let F be the claim size of car insurance

$$E \sim N(800, 100^2)$$
$$F \sim N(1200, 300^2)$$

$$P(F_1 + F_2 + F_3 > E_1 * E_2 + E_3 + E_4 + 800)$$
$$P((F_1 + F_2 + F_3) - (E_1 + E_2 + E_3 + E_4) > 800$$
$$(F_1 + F_2 + F_3) - (E_1 + E_2 + E_3 + E_4) \sim N(3600 - 41800);$$

$$300^2(3) + 100^2(\sim N(400, 31000)$$

$$P\left(Z > \frac{800 - 400}{\sqrt{31000}}\right)$$

$$P(Z > 0.71842) \Rightarrow 1 - P(Z < 0.71842)$$
$$\Rightarrow 1 - P(Z < 0.72)$$
$$1 - 0.76424$$
$$= 0.23576$$

2. $N \sim \text{Neg Bin}(k, p)$

$$P(N = n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$$
$$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$$
$$k = 3, \quad p = 0.9$$

$$M_Y t = M_N(\log M_X t)$$

$$X \sim \text{gamma}(6, 2)$$

$Y \rightarrow$ total claim amount

(i) MGF of Y

$$M_y(t) = E(E(e^{yt} \mid N = n))$$

$$E(e^{yt} \mid N = n) = E(e^{x_1 t + x_2 t + \cdots x_n t})$$

$$= E(e^{t x_1}) \cdot E(e^{t x_2}) \cdots E(e^{t x_n})$$

$$= \left(1 - \frac{t}{2}\right)^{-6n}$$

So, $E(E(e^{yt} \mid N = n)) = E\left(\left(1 - \frac{t}{2}\right)^{-6n}\right)$

$$= E\left(e^{n \log (1-t/2)^{-6}}\right)$$

$$= M_n\left(\log\left(1 - \frac{t}{2}\right)^{-6}\right)$$

$$M_n(t) = \left(\frac{pe^t}{1 - qe^t}\right)^k \Rightarrow \left(\frac{pe^{\log(1-t/2)^{-6}}}{1 - q\, e^{\ln(1-t/2)^{-6}}}\right)^k$$

$$\left[\frac{p\,(1-t/2)^{-6}}{1 - q\,(1-t/2)^{-6}}\right]^3 \Rightarrow \frac{(0.9)\left(1 - \frac{t}{2}\right)^{-6}}{6 - 0.1\left(1 - \frac{t}{2}\right)^{-6}}$$

$$V(y) = E(N)V(x) + (E(x)^2 V(N))$$

$$= \left(\frac{3}{0.9} \times \frac{6}{4}\right) + \left(\frac{3}{2}\right)^2 \times \frac{(3)(0.1)}{(0.9)^2}$$

$$\sigma^2 = 2.887$$

3. $f(x) = \lambda e^{-\lambda(x-\alpha)}$

$\text{MGF} = E(e^{tx}) \Rightarrow \displaystyle\int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} \, dx$

$M_x t = \lambda \displaystyle\int_{\alpha}^{\infty} e^{tx} e^{-\lambda x} e^{\lambda\alpha} \, dx \Rightarrow e^{\lambda\alpha} \lambda \int_{\alpha}^{\infty} e^{-(\lambda-t)x} \, dx$

$\dfrac{e^{\alpha\lambda} \lambda}{t-\lambda} \left[e^{-(\lambda-t)x} \right]_{\alpha}^{\infty}$

$\Rightarrow \dfrac{e^{\lambda\alpha} \lambda}{t-\lambda} \left[-e^{-(\lambda-t)\alpha} \right]$

$\Rightarrow e^{t\alpha} \dfrac{\lambda}{\lambda-t}$

(ii) $M_x(t) = \dfrac{\lambda e^{t\alpha}}{\lambda - t} \Rightarrow \lambda e^{t\alpha} (\lambda-t)^{-1}$

$M_x'(t) = \lambda \left[e^{t\alpha} (\lambda-t)^{-2} (+1) + (\lambda-t)^{-1} e^{t\alpha} (\alpha) \right]$

$= \lambda \left[(\lambda-t)^{-1} e^{t\alpha} (\alpha) + e^{t\alpha} (\lambda-t)^{-2} \right]$

$= \lambda e^{t\alpha} (\lambda-t)^{-2} \left[\alpha(\lambda-t) + 1 \right]$

$t = 0 \Rightarrow \lambda(\lambda)^{-2} \left[\alpha(\lambda) + 1 \right]$

$\Rightarrow \dfrac{1}{\lambda} \left[\lambda\alpha + 1 \right] = \dfrac{\lambda\alpha + 1}{\lambda}$

$E(x^2) = M_x''(t) \Rightarrow \lambda \left[\alpha e^{t\alpha} (\lambda-t)^{-2} + \alpha^2 (\lambda-t)^{-1} e^{t\alpha} \right.$
$\left. + e^{t\alpha} (+2)(\lambda-t)^{-3} \alpha + (\lambda-t)^{-2} e^{t\alpha} \right]$

$t = 0$

$\lambda \left[\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + (2)(\lambda)^{-3} + \alpha(\lambda)^{-3} \right]$

$\lambda \left[\dfrac{2\alpha}{\lambda^2} + \dfrac{\alpha^2}{\lambda} + \dfrac{2}{\lambda^3} \right]$

$E(x^2) = \dfrac{2\alpha}{\lambda} + \alpha^2 + \dfrac{2}{\lambda^2}$

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$$V(x) = E(x^2) - [E(x)]^2$$

$$\frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\alpha + \frac{1}{\lambda}\right]$$

$$\frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda}\right]$$

$$\frac{2}{\lambda^2} - \frac{1}{\lambda^2} \Rightarrow \boxed{\frac{1}{\lambda^2}}$$

4. $G_V(t) = \left(\frac{p}{1-qt}\right)^k$

$$G_V t = \sum t^v \times P(V=v)$$

$$P(V=4) = \frac{\sqrt{K+4}}{\sqrt{5}\sqrt{K}} \cdot p^k q^k$$

$$= \frac{\sqrt{K+4}}{4!\sqrt{K}} p^k q^k$$

$$= \frac{(K+3)!}{(K-1)!\,4!} p^k (1-p)^K$$

5. $f(x,y) = ax(x+y)$

(i) $P\left(y > \frac{1}{3}x + 10\right)$

$$a\int_{10}^{20}\int_0^5 (x^2+xy)\,dx\,dy = 1$$

$$\int_{10}^{20}\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5 dy = \frac{1}{a}$$

$$\int_{10}^{20}\left(\frac{125}{3} + \frac{25}{2}y\right)dy = \frac{1}{a}$$

$$\left[\frac{125y}{3} + \frac{25y^2}{4}\right]_{10}^{20} = \frac{1}{a}$$

$$833.33 + 2500 - (416.667 + 625) = 1/a$$

$$a = \underline{0.00043636}$$

$0 < x < 5, \quad 10 < y < 20$

$$a\left(\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right) = 1$$

$$a\left(\frac{1250}{3} + 1875\right) = 1$$

$$a\left(\frac{6875}{3}\right) = 1$$

$$a = 3/6875$$

(iii) $E(Y/X)$

$$f\,y/x = \frac{f(x,y)}{f(x)}$$

$$f(x) = \frac{3}{6875}\int_{10}^{20} (x^2 + xy)\,dy$$

$$= \frac{3}{6875}\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= \frac{3}{6875}\left[10x^2 + 150x\right]$$

$$= \frac{30x}{6875}\left[x + 15\right]$$

$$f\,y/x = \frac{3}{6875}\,\frac{x(x+y)}{30x} \times 6875(x+15) \Big| \frac{3}{6875}\,x$$

$$f\,y/x = \frac{x+y}{10(x+15)}$$

So, $$E(Y/X) = \int_{10}^{20} y \cdot \frac{(x+y)}{10(x+15)}\,dy$$

$$\frac{1}{10(x+15)}\int_{10}^{20} (xy + y^2)\,dy$$

$$\frac{1}{x+15}\left[15x + \frac{700}{3}\right]$$

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6. $X \sim Poi(\lambda)$ $\quad Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ $\quad M_Y(t) = e^{\mu(e^t-1)}$

$Z = X - Y$

$M_Z(t) = E(e^{tZ}) = E(e^{t(X-Y)}) = E(e^{tX} \cdot e^{-tY})$

$= E(e^{tX}) \times E(e^{-tY})$

$= \dfrac{E(e^{tX})}{E(e^{tY})}$

$= \dfrac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$

$= e^{(\lambda-\mu)(e^t-1)}$

By uniqueness MGF of $Z = e^{(\lambda-\mu)(e^t-1)} \sim Poi(\lambda-\mu)$

Let $Z = X + Y$

$E(e^{tZ}) = e^{\lambda(e^t-1)} \times e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

By uniqueness $Z \sim Poi(\lambda+\mu)$

7.

		X: 1	2	3	4
Y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

(i) $V(X/Y=2) = E(X^2/Y=2) - [E(X/Y=2)]^2$

$E(X^2|Y=2) \rightarrow \sum x^2 \dfrac{P(X=x, Y=2)}{P(Y=2)}$

when $x=1$ $\quad x=2$ $\quad x=3$ $\quad x=4$

$\dfrac{(1)^2 P(X=1, Y=2)}{P(Y=2)} + \dfrac{(2)^2 P(X=2, Y=2)}{P(Y=2)} + \dfrac{(3)^2 P(X=3, Y=2)}{P(Y=2)} + \dfrac{(4)^2 P(X=4, Y=2)}{P(Y=2)}$

$\dfrac{(1)(0)}{6} + \dfrac{(4)(0.3)}{0.6} + \dfrac{(9)(0.1)}{0.6} + \dfrac{16(0.2)}{0.6}$

$= \underline{8.8333}$

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$$E\left(X/Y=2\right) \Rightarrow \sum_x x \frac{P(X=x, Y=2)}{P(Y=2)}$$

$$\frac{1(0)}{0.6} + \frac{(2)(0.3)}{0.6} + \frac{(3)(0.1)}{0.6} + \frac{4(0.2)}{0.6}$$

$$= \frac{0.6 + 0.3 + 0.8}{0.6}$$

$$= 2.8333$$

$$\Rightarrow \boxed{0.807}$$

(ii) $f(u,v) = \frac{48}{67}(2uv - u^2)$ $\quad 0<v<1 \; ; \; \frac{u}{2}<v<2$

$$E(U/V=v) = ?$$

$$\frac{f(u,v)}{f(v)} \rightarrow f\, U/V = U$$

$$f(v) \Rightarrow \frac{48}{67}\int_0^1 (2uv - u^2)\,du$$

$$\frac{48}{67}\left[\frac{2u^2v}{2} - \frac{u^3}{u}\right]_0^1$$

$$= \frac{16}{67}(3v-1)$$

$$f(v) = \frac{16}{67}(3v-1)$$

$$\frac{f(u,v)}{f(v)} = \frac{48}{67} \frac{(2uv^2 - u^2)}{16(3v-1)} \times 67$$

$$\Rightarrow \frac{3}{3v-1}(2uv^2 - u^2)$$

$$E\left[U/V=v\right) = \frac{3}{3v-1}\int_0^1 u(2uv^2 - u^2)\,du$$

$$= \frac{3}{3v-1}\left[\frac{2u^3v^2}{3} - \frac{u^4}{u}\right]_0^1$$

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$$\frac{3}{3v-1}\left[\frac{2v^2}{3}-\frac{1}{4}\right)$$

$$\frac{3}{3v-1}\left[\frac{8v^2-3}{12}\right]$$

$$= \boxed{\frac{8v^2-3}{4(3v-1)}}$$

8. $X \sim \text{gamma}(3,2)$ $\quad E(X) = 3/2 \quad V(X) = 3/4$

$E(Y/X=x) = 3x+1 \qquad var(Y/X=x) = 2x^2+5$

$$E(Y) = E(E(Y/X=x))$$
$$= E(3x+1)$$
$$= 3E(x)+1$$
$$= \frac{9}{2}+1 = \boxed{\frac{11}{2}}$$

$$V(Y) = E(V(X/Y)) + V(E(X/Y))$$
$$= E(2x^2+5) + V(3x+1)$$
$$= 2E(X^2) + 5 + 9V(x)$$
$$V(X) = E(X^2) - [E(X)]^2$$
$$\frac{3}{4} + \frac{9}{4} = E(X^2)$$
$$E(X^2) = 12/4 = 3$$

$$V(Y) = 2(3) + 5 + 9(3/4)$$
$$= 11 + 27/4$$
$$= \boxed{\frac{71}{4}}$$

9. $E(x) = 3$ $\quad V(x) = 4$

$E(y) = 4$ $\quad V(y) = 1$

correlation coeff. $= -0.3\sqrt{4} = -0.6$ $\quad \operatorname{cov}(x, y)$

$z = x + y$

(i) $\operatorname{cov}(x, x+y) = \operatorname{cov}(x, x) + \operatorname{cov}(x, y)$

$= V(x) + \operatorname{cov}(x, y)$

$= 4 + (-0.6)$

$= 3.4$

(ii) $\operatorname{var}(z) = \operatorname{cov}(x+y, x+y)$

$= \operatorname{cov}(x, x) + 2\operatorname{cov}(x, y) + \operatorname{cov}(y, y)$

$= V(x) + 2\operatorname{cov}(x, y) + V(y)$

$= 4 + 2(-0.6) + 1$

$= 5 - 1.2$

$= 3.8$

10. $f(x, y) = k x^{-\alpha} e^{-y/\beta}$

$1 < x < \infty \quad 1 < y < \infty$

$\alpha > 1, \ \beta > 0$

$$k \int_1^{\infty} \int_1^{\infty} x^{-\alpha} e^{-y/\beta} \, dx \, dy = 1$$

$$k \int e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1} \right]_1^{\infty} = 1$$

$$\frac{k}{1-\alpha} \int e^{-y/\beta} [-1] = 1 \Rightarrow \frac{k}{\alpha-1} \int e^{-y/\beta} dy = 1$$

$$\frac{k}{\alpha-1} \frac{\left[e^{-y/\beta}\right]_1^{\infty}}{1/\beta}, \quad \frac{-k}{\alpha-1}(\beta)(e^{-1/\beta}) = 1$$

$$K = \frac{\alpha - 1}{(\beta)(e^{-1/\beta})}$$