-ARYAN TRIPATHI

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FE → Assignment 1 (Unit 1 and 2)

1) i) an arbitrage opportunity is a situation where the investor can make a risk-free profit because of the price mismatch of the same underlying asset in 2 diff. markets.
∴ the prob. of loss is 0 & that of earning a profit is +ve.

ii) It states that any 2 portfolios that have exactly the same weights must have the same price. This is the Law of one price. If this is not true then the investor can buy at a cheaper price & sell at a higher price to earn a riskless profit.

iii) $K = 120$ (put optn) ; $S_0 = 125$, $r = 5\%$
$t = 3/12$; call optn $= 30$

a) $q = 15\%$ p.a.

$$p = c_t + Ke^{-r(T-t)} - S_t e^{-q(T-t)}$$
$$= 30 + 120\, e^{-5\% \times 0.25} - 125\, e^{-15\% \times 0.25}$$
$$p = 28.11$$

b) If $p = 23$.
Investor can buy a put at a cheaper price & sell a call at a higher price.

sell call : 30 — Borrow 100 to buy asset.
buy put : 23. — buy 1 share at 125

Put call parity doesn't hold true. investor can make arbitrage profit.

2) $dX_t = \alpha r(T-t)\,dt + \sigma\sqrt{(T-t)}\;dZ_t$

$dF = -\int B_t\, dZ_t + \int \frac{dm}{dt}(T-t) - \int A_t\, dt + \frac{1}{2}\int B_t^2\, dt$

$$df = f\left(\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}B_t^2\right)dt - fB_t + dZ_t$$

For f to be a martingale,

$$\frac{dm}{dt}(T-t)^{-A_t} + \frac{1}{2}B_t^3 = 0$$

$$\therefore \frac{dm}{dt}(T-t) = A_t + \frac{1}{2}B_t^2$$

$$\frac{dm}{dt}(T-t) = \alpha u - \frac{1}{2}\sigma^2 \quad \text{(from I)}$$

(3) $F(t,x) = e^{-t}x^2$

x is SDE

$$\frac{dX_t}{X_t} = 0.25dt + \sigma dW_t$$

$$\frac{df}{dt} = -e^{-t}x^2 = -b$$

$$\frac{df}{dx} = e^{-t}2x$$

By ito's lemma

$$dY_t = \frac{df}{dt}\cdot dt + \frac{df}{dx}\cdot dX_t + \frac{1}{2}\frac{d^2 f}{dx^2}\cdot\sigma^2 x^2 dt$$

$$= -Y_t\,dt + 2e^{-t}X_t^2\,\frac{dX_t}{X_t} + e^{-t}\sigma^2 X_t^2\,dt$$

$$= -Y_t\,dt + 2Y_t\left[0.25dt + \sigma dW_t\right] + \sigma^2 Y_t dt$$

$$\therefore \frac{dY_t}{Y_t} = \left[2u - \tfrac{1}{4} + \sigma^2\right]dt + 2\sigma dW_t$$

$$= \left[\sigma^2 - 0.5\right]dt + 2\sigma\,dW_t$$

$$\Rightarrow dY_t = \left[\sigma^2 + 0.5\right]Y_t\,dt + 2\sigma Y_t\,dW_t$$

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ii) The process is a martingale if drift is 0
This means $\sigma^2 - 0.5 = 0$, $\sigma^2 = 0.5$

(4) dividends are d_1, d_2 & d_m.

opt^n exercised before ex-dividend date $= S_{(t)} - K$
opt^n not exercised, price drops to $S_{(t)} - d$

The value of the american opt^n is greater that $S_{(t)} - d - Ke^{-r(T-t)}$

If is never optimal to exercise the opt^n if

$S_{(t)} - d_n - Ke^{-r(T-t)} \geq S_{(t_n)} - K$

using this, we have
$K(1-e^{-r(T-t)}) = 350(1-e^{-0.95(0.83-0.25)})$
$= 18.87$
and $65(1-e^{-0.95(0.83-0.25)}) = 10.91$

(5) $P(S > 258) = ?$ S is GBM

$\Rightarrow S_t = S_0 \exp(r - \sigma^2/2)t + \sigma W_t$
$\therefore \ln(S_t) \sim N((\ln(S_0) + (r - \sigma^2/2)t + \sigma W_t), \sigma^2 t)$

$\therefore \ln(S_t) \sim \phi(\ln 254 + (0.16 - 0.35^2/2) 0.5, 0.35(0.5)^{1/2})$

$= \phi(5.59, 0.247)$
$\therefore P(S > 258) = 1 - N(5.55 - 5.59)/0.247$
$= 1 - N(0.1564)$
$= 0.5542$
Put opt^n exercised if stock prices $<$ 258 in 6 months
$prob = 1 - 0.5542 = 0.4457$

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(6) $\ln\left(\frac{S_t}{S_0}\right) = ut + \sigma B_t$

$\frac{S_t}{S_0} = e^{ut + \sigma B_t}$

$y = S_t = S_0 \, e^{ut + \sigma B_t}$

(i) $\frac{dS_t}{S_t} = x \, dB_t + y \, dt$

By Itô's Lemma,

$dS_t = u S_t \, dt + \sigma S_t \, dB_t + \sigma^2 \frac{S_t}{2} dt$

$= S_t . dt \left(u + \frac{\sigma^2}{2}\right) + \sigma S_t . dB_t$

$\frac{dS_t}{S_0} = x \, dB_t + y \, dt$

$x = \sigma, y = u + \frac{1}{2}\sigma^2$

(ii) $E(S_t) = E(S_0 \, e^{ut + \sigma B_t}) = S_0 e^{ut} E(e^{\sigma B_t})$

$= S_0 e^{t(u + \frac{1}{2}\sigma^2)}$

$Var(S_T) = E(S_T^2) - [E(S_T)]^2$

$= E(S_0^2 \, e^{2ut + 2\sigma B_t}) - (S_0 e^{t(u + \frac{1}{2}\sigma^2)})^2$

$= S_0^2 . e^{2ut} . E[e^{2\sigma B_t} - S_0 e^{t(u + \frac{1}{2}\sigma^2)})^2]^2$

$= S_0^2 e^{2ut} (e^{2\sigma^2} - e^{\sigma^2 t})$

(iii) $Cov(S_{t_1}, S_{t_2}) = E(S_{t_1}, S_{t_2}) - E(S_{T_1})(S_{T_2})$

$\rightarrow E(S_{t_1} S_{t_2}) : S_0 e^{ut_1} E(e^{\sigma B_{t_1}}) S_0 e^{ut} E(e^{\sigma B_{t_2}})$

$\rightarrow S_0^2 e^{u(t_1 + t_2)} \quad E[e^{\sigma B_{t_1} + \sigma(B_{t_1} + (B_{t_2} - B_{t_1}))}])$

expectation is sum of the MGF of normal distⁿ.

$= S_0^2 \, e^{u(t_1+t_2)} \cdot e^{S_2 \sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2}$

(8) (i) portfolio ∴ short position in the option & long position in Δ shares

↑ movmt. in port. value $= \Delta S_0 u - f_u$

if ↓ movmnt. the value of port. $= \Delta S_0 d - f_d$.

2 portfolios are equal if:

$\Delta S_0 u - f_u = \Delta S_0 d - f_d$

PV of portfolio $= (\Delta S_0 u - f_u) e^{-rT}$

Cost of the 2 portfolios: is $\Delta S_0 - f$

port. grows at r.f. rate.

$\Delta S_0 - f = (\Delta S_0 u - f_u) e^{-rt}$

$f = e^{-rT} [p f_u + (1-p) f_d]$

where $p = [e^{rT} - d] / [u - d]$

(ii) The option pricing formula does not involve the prob. of stock going ↑ or ↓, but it is natural to assume that the prob. of an upward stock movement increases the value of a call optⁿ & ↓ in value of put optⁿ when the prob. of stock price goes down.

This is bc. we are calculating the value of the optⁿ in terms of the value of underlying stock. It is natural to assume p as prob. of up. movement & $1-p$ for down movement.

Thus, the above eqn can be interpreted as the value of the optⁿ today is the expected future value discounted at the r.f. rate.

(iii) $E(S_T)$ at time $T = p S_0 u + (1-p) S_0 d \cdot 0.5$

substituting p from above eqⁿ in part (i)

$$E(S_T) = e^{rT} S_0$$

⇒ stock price grows at r.f. rate.

(iv) In a risk-neutral world, individuals do not require compensation for risk. Hence the exp. return on all securities and optⁿ is the r.f. rate. Hence value of optⁿ is the exp. payoff in a risk ~~free~~ neutral set up discounted at r.f. rate.

(a) forward price $= S \exp(A)$
Put-call parity $= C + Ke^{-rt} = p + S$

substituting values for simult. eqⁿ.

$$13.334 + 70 \cdot \exp(-0.25r) = 0.120 + S$$
$$8.869 + 75 \cdot \exp(-0.25r) = 0.568 + S$$

$\Rightarrow F = 83.45.$

(ii)
roi for the next 6 months is r_6
$\exp(r_6 \times 0.5) = \exp(r_3 0.25) \exp \cdot (r_f \times 0.25)$
$r_3 = 7.1$.

for r_6 : $2.569 + 90 \exp(-0.5 r_6) = 7.909 + 82$
$r_6 = 6.1$ & $r_f = 5.1$

(iii) put-call parity for each row.
$$6.899 + 9 \exp(-7.1 \times 0.25) = 1.055 + 82$$
$$b + 80 \exp(-7.1 \times 0.25) = 1.789 + 82$$

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$2.394 + 85\exp(-7\% \times 0.25) = c + 82$

after solving:

$a = 77.5$

$b = 5.177$

$c = 4.119$

10 f.o. interest are assumed zero, the risk-neutral prob.

(i) is:

$$q = (1-d) / (u-d)$$

for a recombining tree, $d = 1/u$

$$\therefore q = \left(1 - \frac{1}{u}\right) \Big/ \left(u - \frac{1}{u}\right) = \frac{1}{u+1}$$

for no arbitrage, we must have $u > 1 > d$

Then $u > 1 \Rightarrow u+1 > 2$

$$q = \frac{1}{u+1} < \frac{1}{2}$$ hence proved.

(ii) As derived for (i), $q = 1/_{u+1}$

$q = \frac{1}{3} \Rightarrow u = 2$

we know that $u = \exp(\sigma\sqrt{\Delta t})$

using $u = 2$ & $\Delta t = \frac{1}{12}$

$\sigma = 240.11$

(iii) $n = 12. \Rightarrow q = 1/3$

$u = 2$

$d = 1/2.$

derivative has a payoff $\sqrt{\frac{S_T}{S_0}}$ at time $T = 12$ months

$$\text{The CP} = P = \sum_{k=0} \sqrt{\frac{S_0 u^k d^{n-k}}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$$

$= 0.49327.$

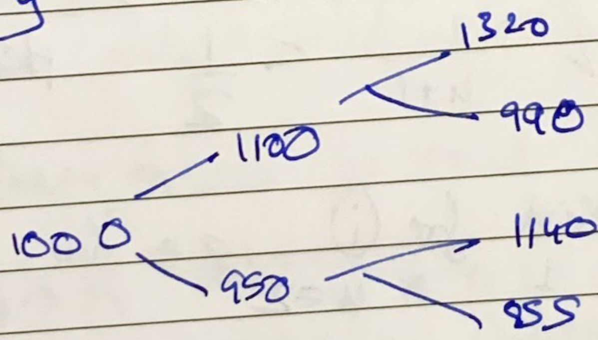
(ii) (a) In a recombining binomial tree is one in which the sizes of up and down steps is the same under all the states & across all time intervals.

It ∴ follows that the risk neutral probab 'q' is always constant at all times & in all states.

The main adv. is that it only has (n+1) possible states at t time as opposed to 2^n possible states in a non-recombining tree.

The main disadv. is that it implicitly assumes the volatility & drift parameters of the underlying are constant over time.

(b) i)



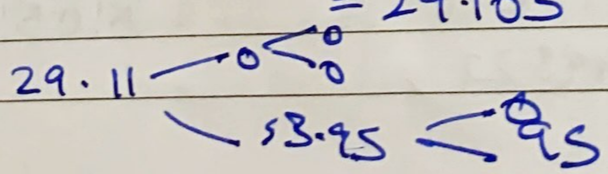
$$q_1 = \frac{e^{(0.0175)} - 0.95}{110 - 0.95} = 0.4510$$

$$q_2 = \frac{e^{0.025} - 0.90}{1.20 - 0.90} = 0.41772$$

4 states: 0, 0, 0, £095

current value = exp (0.0175) = (0.4510 × 0) + (0.5490 × 53.9508)

= 29.105



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(iii) 2^{nd} proposed modification would produce more accurate valuation, a lot more parameters would have to be specified.

New tree would be $2^6 = 64$ nodes & computationally cumbersome.

(12) (i) $Z(t)$ is a std. Brownian

a) $du(t) = 2dZ - 0$
$= 0dt + 2dZ_t$
Thus, $U(t)$ has zero drift

b) $dV(t) = d[Z(t)]^2 - dt$
$d[Z(t)]^2 = 2Z(t)\, dZ_t + dt$

Thus, $dV(t) = 2Z_t\, dZ_t$. The stochastic process $V(t)$ has 0 drift

c) $dW(t) = d[t^2 Z(t)] - 2t\, Z(t)\, dt$
because $d[t^2 Z(t)] = t^2 dZ(t) + 2t\, Z(t)\, dt$

$dW(t) = t^2\, dZ(t)$
Thus, the process $W(t)$ has 0 drift.

(13) $\frac{S_t}{S_0} \sim LN\left[(\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right]$

expected return - μ
volatility $= \sigma$

$S_0 e^{\nu \delta t} = q S_0 u + (1-q) S_0 d$
$q = (e^{\mu \delta t} - d) | (u-d)$
s.d of stock price $= \sigma\sqrt{\delta t}$

The variance is:

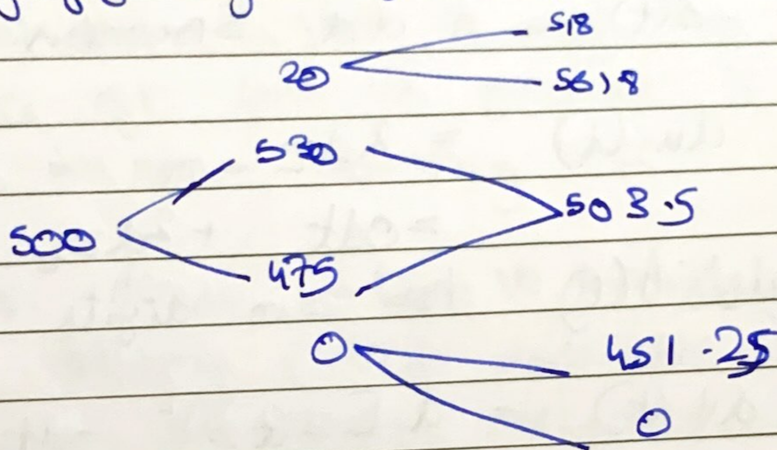
$$q\,U^2 + (1-q)\,d^2 - [qu + (1-q)d]^2$$

$$\therefore \quad e^{u\delta t}(u+d) - ud - e^{2u\delta t} = \sigma^2 dt$$

$$u = e^{\sigma\sqrt{\delta t}} \text{ and } d = e^{-\sigma\sqrt{\delta t}}$$

(14) Payoff Diagram for European call

(i)



$u = 1.06$

$d = 0.95$

$r = 0.05$

$t = 0.25$

$p = \exp(rT) - d \,/\, (U-d) = 0.5689$

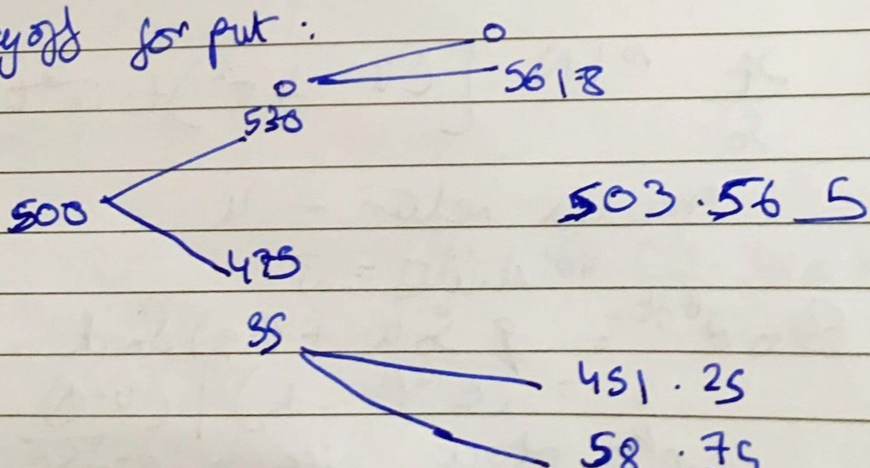
$1-p = 0.4311$

Value of 6 month European call: $\exp(-0.5 \times 0.05)$

$= 51.8 \times 0.5689^3$

$= 16.351$

(ii) Payoff for put:



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value of European put, $\exp(-0.05 \times 0.5) \times$
$[0.4311^2 \times 58.75 + 2 \times 6.04311 \times 0.5689 \times 6.50]$
$= 13.759$

using put-call parity

value of put + stock $= 513.759$
value of call + discounted price $= 513.759$

(iii) For american opt^n, the value at final rate is same for a European opt^n.

value at node B $= 6.50 \exp(-0.25 \times 0.05) \times 0.4311$
$= 2.767$
$C = \exp(-0.25 \times 0.05)(6.50 + 0.5689) + 58.75 \times 0.4311$

value of American put option at $t=0$
$\exp(-0.05 \times 0.25)(35 \times 0.4311 + 2.767 + 0.5689)$
$= 16.46$

(iv) real rate of return $= 9\%$
$p = 0.6614$
Exp. payoff $= 20 \times 0.6614$
$= 13.228$

(15) (i) Let $b = b(S_t, t) = S^k$

$\frac{db}{dS} = kS^{k-1}$, $\frac{d^2 b}{dS^2} = k(k-1)S^{k-2}$

& $\frac{db}{dt} = 0$

$df = kS^{k-1} dS_t + \frac{1}{2} k(k-1) S^{k-2} (dS_t)^2$

$= f[ku + \frac{1}{2} k(k-1)\sigma^2] dt + f(k\sigma) dW_t$

Hence $f = S^k$ in GBM
w/ drift $= ku + \frac{1}{2} k(k-1)\sigma^2$

volability $= k\sigma$

$f_t = f_0 e^{(u - 1/2 \sigma'^2)t + \sigma' W_t}$

This means $\frac{f}{f_0} \sim LN[(u' + \frac{1}{2}\sigma'^2)t, \sigma'^2 t]$

$E(f) = f_0 e^{u't}$, $V(f) = f_0 e^{2u't}(e^{\sigma'^2 t} - 1)$

(ii) Let $f = f(t, S_t) = e^{-rt} S_t$

$\frac{\partial f}{\partial t} = -re^{-rt} S_t$, $\frac{\partial f}{\partial S} = e^{-rt}$, $\frac{\partial^2 f}{\partial S^2} = 0$

by Ito calculus,
$df = -re^{-rt} S_t dt + e^{-rt} dS_t$

$= (u - r) f dt + \sigma f dW_t$

Which is a martingale if & only if $u = r$

(iii) S^k to be a martingale $= ku + \frac{1}{2} k(k-1)\sigma^2 = r$