Assignment -1 Financial Engineering Atharva Padwal

Q1.

i) An arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as free lunch. An arbitrage opportunity means that
→ we ~~can~~ start at time 0 with a portfolio that has a net value of zero. (implying that we are long in some assets & short in others)
– at some future time T:
probability of loss is 0
Probability that we make a strictly positive profit is greater than 0

ii) Law of one price is an economic theory that states the price of identical goods in different markets must be the same after taking the currency exchange into consideration.

iii) ~~Current~~ $S_t = 125$ $C_t = 30$ $q = 15\%$.
$K = 120$ $r = 5\%$.

$$P_t = \cancel{30+} C_t + Ke^{-r(T-t)} - S_t e^{-q(T-t)}$$

$$= 30 + 120e^{-0.05(0.25)} - \cancel{\$}\ 125\, e^{-0.15 \cdot 0.25}$$

$$= ₹ 28.11$$

If the put option are only ₹ 23, then they are cheap. To generate ⇒ profit, buy cheap & sell expensive i.e. buy put & share & sell call & cash.

For example.

sell 1 call at ₹ 30

buy 1 put at (₹ 23)

buy 1 share at (125)

sell (borrow) cash at ₹ 118

This is a zero cost portfolio and because put call parity does not hold, we know it will make an arbitrage profit.

In 3 months,

the repaying cost of cash

$= 118\ e^{0.05 \times 0.25}$

$= 119.48$

We will also receive dividends, totalling d.

1. If $S_T > 120$ ~~this at~~ then call is exercised by other party and profit ⇒ $120 - 119.48 + d$

⇒ $0.52 + d$

2. If $S_T < 120$, then put will be exercised by us.

Profit $120 - 119.48 + d = 0.52 + d$.

In both cases, zero cost portfolio generates positive future profits.

3.

i) $F(t, x) = e^{-t} x^2$

$\frac{df}{dt} = -e^{-t} x^2 = -f$

$\frac{df}{dx} = 2e^{-t} x$

$\frac{df^2}{dx^2} = 2e^{-t}$

$Y_t = f(t, x_t)$

Using ito's lemma

$dY_t = \frac{df}{dt} dt + \frac{df}{dx} dx_t + \frac{1}{2} \frac{d^2 f}{dx^2} \sigma^2 x_t^2 dt$

$= -f dt + 2e^{-t} x_t dx_t + e^{-t} \sigma^2 x_t^2 dt$

$= -Y_t dt + 2e^{-t} x_t^2 \frac{dx_t}{x_t} + e^{-t} \sigma^2 x_t^2 dt$

$= -Y_t dt + 2Y_t [0.25 dt + \sigma dw_t] + \sigma^2 Y_t dt$

$\frac{dY_t}{Y_t} = -dt + 0.5 dt + 2\sigma dw_t + \sigma^2 dt$

$= dt[-1 + 0.5 + \sigma^2] + 2\sigma dw_t$

$dY_t = (\sigma^2 - 0.5) \cancel{dt} Y_t dt + 2\sigma Y_t dw_t$

ii) 2. Process is a martingale if drift $= 0$

i.e $\sigma^2 - 0.5 = 0$

$\therefore \sigma^2 = 0.5$

4. Let n ex dividend dates are anticipated for a stock and $t_1 < t_2 < t_3 < \dots < t_n$ are the times before which the stock goes ex-dividend.
Dividends are denoted by $d_1 \dots d_n$
If the option is not exercised, Price drops to $S(t_n) - d_n$
If the option is exercised prior to the ex-dividend date then the investor receives $S(t_n) - K$
The value of the american option is greater than
$S(t_n) - d_n - K\exp(-r(T-t_n))$.
It is never ~~optimal~~ ~~optional~~ optimal to exercise the option
if $S(t_n) - d_n - K\exp(-r(T-t_n)) >= S(t_n) - K$
i.e.

$$d_n <= K*(1-\exp(-r(T-t_n))$$

Using this equation:

$$K*(1-e^{-r(T-t_n)}) = 350*(1-e^{-0.095*(0.8333-0.25)})$$
$$= 18.87$$
$$65*(1-e^{-0.95*(0.8333-0.25)}) = 10.91$$

Hence it is never optimal to exercise the american option on the two ex-dividend dates

5. The reqd. probability is the prob of the stock price being greater than ₹258 in 6 months

$$S_t = S_0 e^{(u-\sigma^2/2)t + \sigma W_t}$$

$$\therefore \ln(S_t) \sim N(\ln(S_0) + (u+\sigma^2)t, \sigma^2 t)$$
$$\therefore \ln(S_t) \sim \varphi(\ln 254 + (0.16 - 0.35^2)*0.5, 0.35 \times 0.5)$$
$$= \varphi(5.59, 0.247)^2$$

This means that $\dfrac{[\ln(S_t) - \mu(S_t)]}{\sigma t^{1/2}} \sim$ ~~SND~~ $N(0,1)$

$\therefore$ Prob(~~stock will be~~ $S_{\cancel{t}} > \overset{\overset{K}{\downarrow}}{258}$)

$= 1 - N\left(\dfrac{5.55 - 5.59}{0.247}\right) = 1 - N(-0.1364)$

$= 0.5542$

Put option will be excercised with a probability of $1 - 0.5542 = \underline{\underline{0.4457}}$

Q6. $S_t = S_0 e^{\mu t + \sigma B_t}$ $\qquad dB_t = 0 \times dt + 1 \times dB_t$

Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B_t}$

$\dfrac{\partial G}{\partial t} = \mu S_0 e^{\mu t + \sigma B_t} = \mu S_t$

$\dfrac{\partial G}{\partial B_t} = \sigma S_0 e^{\mu t + \sigma B_t} = \sigma S_t$

$\dfrac{\partial^2 G}{\partial B_t^2} = \sigma^2 S_0 e^{\mu t + \sigma B_t} = \sigma^2 S_t$

Hence using Ito's lemma,

$dG = [0 \times \sigma S_t + \frac{1}{2} 1^2 \times \sigma^2 S_t + \mu S_t] dt + 1 \times \sigma S_t dB_t$

i.e. $dS_t = (\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dB_t$

$\therefore x = \sigma, \quad y = \mu + \frac{1}{2}\sigma^2$

ii)

$E[S_t] = E[S_0 e^{\mu t + \sigma B_t}] = S_0 e^{\mu t} E[e^{\sigma B_t}]$

& $B_t \sim N(0,t)$, i.e. MGF is $E[e^{\theta B_t}] = e^{\frac{1}{2}\theta^2 t}$

$\therefore E[S_t] = S_0 e^{\mu t} \times e^{\frac{1}{2}\sigma^2 t} = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$

$V[S_t] = E[S_t^2] - (E[S_t])^2$

$= E[S_0^2 e^{2\mu t + 2\sigma B_t}] - [S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}]^2$

$= S_0^2 e^{2\mu t} E[e^{2\sigma B_t}] - S_0^2 e^{2\mu t + \sigma^2 t}$

$= S_0^2 e^{2\mu t + 2\sigma^2 t} - S_0^2 e^{2\mu t + \sigma^2 t}$

$= S_0^2 e^{2\mu t}(e^{2\sigma^2 t} - e^{\sigma^2 t})$

iii. $Cov(S_{t_1}, S_{t_2}) = E[S_{t_1} S_{t_2}] - E[S_{t_1}]E[S_{t_2}]$

$E[S_{t_1}] = S_0 e^{\mu t_1 + \frac{1}{2}\sigma^2 t_1}$, & $E[S_{t_2}] = S_0 e^{\mu t_2 + \frac{1}{2}\sigma^2 t_2}$

$E[S_{t_1} S_{t_2}] = E[S_0 \exp(\mu t_1 + \sigma B_{t_1}) \cdot S_0 \exp(\mu t_2 + \sigma B_{t_2})]$

$= S_0^2 e^{\mu(t_1+t_2)} E[\exp(\sigma B_{t_1} + \sigma B_{t_2})]$

To evaluate this we need to split B_{t_2} into two independent components.

$B_{t_2} = B_{t_1} + (B_{t_2} - B_{t_1})$ where $(B_{t_2} - B_{t_1}) \sim N(0, t_2 - t_1)$

$$E[S_{t_1} S_{t_2}] = S_0^2 e^{\mu(t_1+t_2)} E[\exp(\sigma B_{t_1} + \sigma(B_{t_1} + (B_{t_2}-B_{t_1})))]$$

$$= S_0^2 e^{\mu(t_1+t_2)} E(\exp(2\sigma B_{t_1} + \sigma(B_{t_2}-B_{t_1})))$$

$$= S_0^2 e^{\mu(t_1+t_2)} E[\exp(2\sigma B_{t_1})] E[\exp(\sigma(B_{t_2}-B_{t_1}))]$$

$$= S_0^2 e^{\mu(t_1+t_2)} \exp(2\sigma^2 t_1) \exp(1/2\,\sigma^2(t_2-t_1))$$

$$= S_0^2 e^{\mu(t_1+t_2)} e^{3/2\,\sigma^2 t_1 + 1/2\,\sigma^2 t_2}$$

$$\mathrm{CoV} = S_0^2 \mu^{(t_1+t_2)} e^{3/2\,\sigma^2 t_1 + 1/2\,\sigma^2 t_2} - S_0 e^{\mu t_1 + 1/2\,\sigma^2 t_1} \cdot S_0 e^{\mu t_2 + 1/2\,\sigma^2 t_2}$$

$$= S_0^2 e^{\mu(t_1+t_2)} \left(e^{3/2\,\sigma^2 t_1} - e^{1/2\,\sigma^2 t_1}\right) e^{1/2\,\sigma^2 t_2}$$

Q7.

$$\begin{tikzcd} & S_0 u \\ S_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & S_0 d \end{tikzcd}$$

$$E(C_t) = S_0(pu + (1-p)d)$$

$$\mathrm{Var}(C_t) = E(C_t^2) - E(C_t)^2$$

$$= S_0^2[pu^2 + (1-p)d^2] - S_0^2[pu + (1-p)d]^2$$

$$= S_0^2[pu^2 + (1-p)d^2 - (pu + (1-p)d)^2]$$

$$= S_0^2[p(1-p)u^2 + p(1-p)d^2 - 2p(1-p)]$$

$$= S_0^2\, p(1-p)(u-d)^2$$

$$S_0 e^{rT} = S_0[pu + (1-p)d]$$

& $\cancel{\sigma^2 S_0^2 t = S_0[pu + (1- \ldots}$

$$\sigma^2 S_0^2 t = S_0^2\, p(1-p)(u-d)^2$$

$$p = \frac{e^{rt} - d}{u-d}$$

Substituting p in eqn 2

$$\sigma^2 t = \frac{e^{rt}-d}{u-d}\left(1 - \frac{e^{rt}-d}{u-d}\right)(u-d)^2$$

$$= \cancel{(-e^{rt}-d)} - (e^{rt}-d)(e^{rt}-u)$$

$$= (u+d)e^{rt} - (1+e^{2rt})$$

Putting $d = \frac{1}{u}$ and multiplying through by u we get

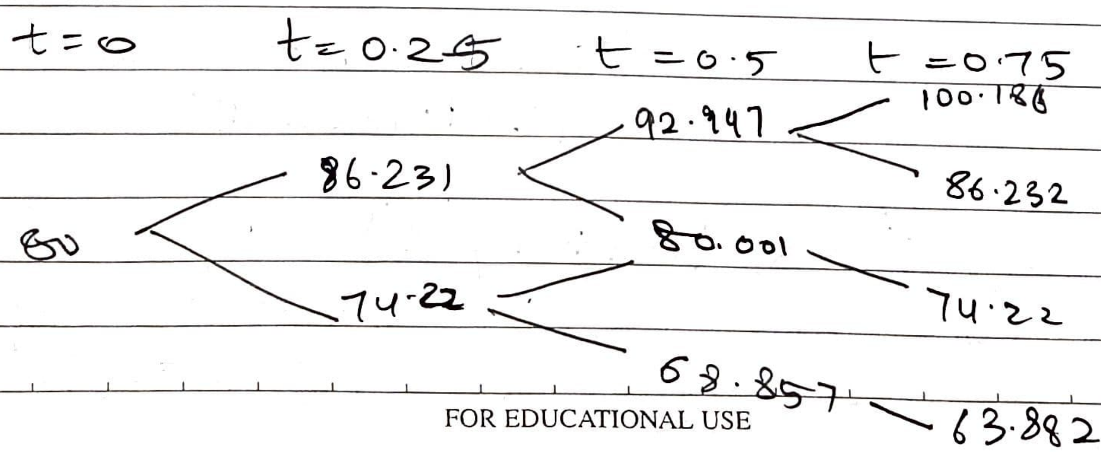
$$u^2e^{rt} - u(1+e^{2rt}+\sigma^2 t) + e^{rt} = 0$$

ii)

a) $\sigma = 0.15$, $t = 0.25 \Rightarrow u = \exp(0.15 * \sqrt{.25}) = \exp(0.075)$
$= 1.077884$

$d = \frac{1}{u} = 0.92774$

∴ Tree



b) $r=0$, we have $p = \frac{e^{rt}-d}{u-d} = \frac{(1-0.927744)}{(1.077884-0.927744)} = 0.4812$

$t=0$	$t=0.25$	$t=0.5$	$t=0.75$
			20.186
		12.948	
	7.787		6.232
4.496		2.9999	
	1.443		0
		0	
			0

c) The lookback call pays the difference between the minimum value and the final value

UUU	20.186
UDU	6.232
UUD	1.232
UDD	0
DUU	12.012
DUD	0
DDU	5.363
DDD	0

Tree value

Node A $= p^3 = .11147$ $= .11147 \times 20.186$

Node B $= p^2(1-p) = 0.12015$ $+ .12015 \times [6.232 + 6.232 + 12.012]$

Node C $= p(1-p)^2 = 0.12950$ $+ 0.12950 \times [5.363]$

Node D $=$ $= 5.8854$

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9 ~~8~~.

i) The forward price is given by, $F = S\exp(rt)$ where S is the stock price, t is the delivery time & r is the continuously compounded risk free-rate of interest applicable up to time t.

Put call parity states : $c + Ke^{-rt} = p + S$ where c & p are the prices of a European call and put option respectively with strike K and time to expiry t & S is current stock price.

$$13.334 + 70.\exp(-0.25r) = 0.12 + S$$
$$8.869 + 75.\exp(-0.25r) = 0.568 + S$$

$S = 82 \qquad r = 7\%.$

$$\therefore F = 82.e^{0.07 \times 0.25} = 83.45$$

ii) Let the rate over next k months be r_k

$$\therefore e^{r_6 \times 0.5} = e^{r_3 \times 0.25} \times e^{r_F \times 0.25}$$

$r_3 = 7\%.$

$$2.569 + 90\, e^{-0.5 \times r_6} = 7.909 + 82$$

$r_6 = 6\%, \; r_F = 5\%.$

iii) $6.899 + a \times e^{-0.07 \times 0.25} = 1.055 + 82$

$b + 80 \times e^{-0.07 \times 0.25} = 1.789 + 82$

$2.594 + 85 \times e^{-0.07 \times 0.25} = c + 82$

$a = 77.6 \;, b = 5.177 \;, c = 4.119$

10.

i. Since interest rates are assumed to be 0, the risk neutral up step probability

$$q = \frac{1-d}{u-d}$$

u & d are up step & down step

For a recombining tree, $d = \frac{1}{u}$

~~Set~~ $q = \frac{1-\frac{1}{u}}{u-\frac{1}{u}} = \frac{1}{u+1}$

For no arbitrage, $u > 1 > d$

$\therefore u+1 > 2 \Rightarrow q = \frac{1}{u+1} < \frac{1}{2}$ Hence proved.

$q = \frac{1}{u+1}$

$q = \frac{1}{3}$, $u = 2$

$u = ~~exp~~\ e^{\sigma\sqrt{\Delta t}}$ $\Delta t = \frac{1}{12}$

$\sigma = 240.11\%$

ii. Since each step is one month and the expiry of the derivative is one year from now.

derivative has payoff $\sqrt{\frac{S_T}{S_0}}$ at $T = 12$ month

$$P = \sum_{k=0}^{n} \sqrt{\frac{S_T}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$$

$$\therefore P = \sum_{k=0}^{n} \sqrt{\frac{S_0 u^k d^{n-k}}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$$

$$= \sum_{k=0}^{n} \sqrt{u^k d^{n-k}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$$

$$= \sum_{k=0}^{n} u^{k/2} d^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$$

$$= \sum_{k=0}^{n} 2^{k/2} \left(\frac{1}{2}\right)^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n} 2^{k-n/2} \frac{n!}{k!(n-k)!} \cdot \frac{2^{n-k}}{3^n}$$

$$= \sum_{k=0}^{n} 2^{n/2} \frac{n!}{k!(n-k)!} \frac{1}{3^n}$$

$$= 2^{n/2} \frac{1}{3^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}$$

$$P = 2^{n/2} \frac{1}{3^n} 2^n = \left(\frac{2\sqrt{2}}{3}\right)^n = \left(\frac{2\sqrt{2}}{3}\right)^{12}$$

$$= 0.49327$$

Q 11

i) A recombining binomial tree is one in which the sizes of the up-steps & down steps are assumed to be the same under all ~~steps.~~ states and all intervals of time.

$\therefore$ ~~It follows~~ risk neutral prob q is also a constant at all times and in all states eg $q_t(j) = q$

The main advantage of 'n' period recombining binomial tree is that it has only $[n+1]$ po

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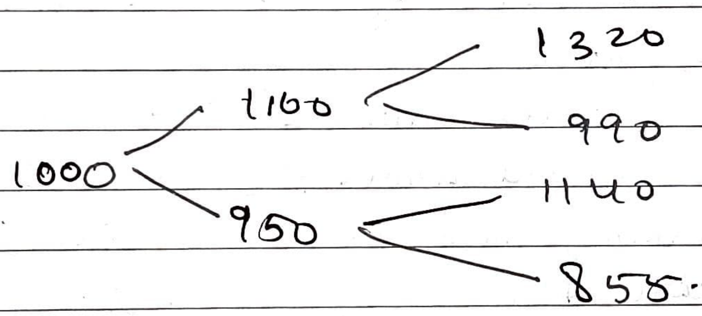
possible states of time as opposed to 2^n possible states in a similar non-recombining tree.
This greatly reduces the amount of computation time required when using binomial tree model.
The main disadvantages is that the recombining binomial tree implicitly assumes that volatility and drift parameters of the underlying asset are constant over time which is contradicted by empirical evidence

ii)
a)



$$q_1 = \frac{e^{0.0175} - 0.95}{1.10 - 0.95} = 0.4510$$

$$q_2 = \frac{e^{0.025} - 0.90}{1.20 - 0.90} = 0.41772$$

Put payoffs $0, 0, 0$ & 95.

$$V_1(1)\, e^{0.025} = [0.41772 \times 0] + (0.58228 \times 0)$$

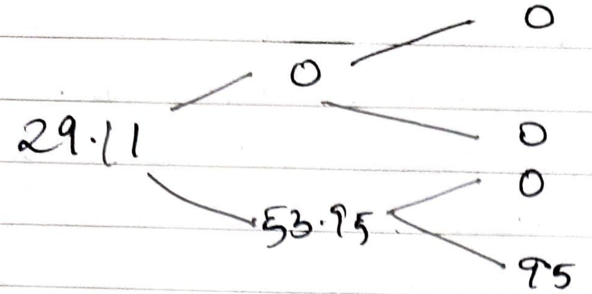
$$V_1(1) = 0$$

$$V_1(2)\, e^{0.025} = (0.41772 \times 0) + (0.58228 \times 95)$$

$$V_1(2) = 53.9508$$

$$V_0 \exp(0.0175) = (0.4510 \times 0) + (0.5490 \times 53.950)$$

$$V_0 = 29.105$$



b) While the proposed modification would produce a more accurate valuation, there would be alot more parameter values to specify. Appropriate values of u & d would be required for each branch of the tree and values of 'r' for each month would be required.

The new tree would have $2^6 = 64$ nodes in the expiry column. This would render the calculations prohibitive to do normally, and would require more programming and calculation time on the computer.

An alternative model that might be more efficient numerically would be a 6-step recombining tree which would have only 7 nodes in the final column.

12. Given $Z(t)$ is standard Brownian

a) $dU(t) = 2\,dZ(t) - 0$

$= 0\,dt + 2\,dZ(t)$.

$\therefore$ Zero drift.

b) $dV(t) = d[Z(t)]^2 - dt$

$d[Z(t)]^2 = 2Z(t).dZ(t) + \frac{2}{2}\, d F[Z(t)]^2$

$= 2Z(t).dZ(t) + dt$

Thus, $dV(t) = 2Z(t)\,dZ(t)$. The process has 0 drift.

c) $dW(t) = d[t^2 Z(t)] - 2tZ(t)\,dt$

Because $d[t^2 Z(t)] = t^2 dZ(t) + 2tZ(t)\,dt$, we have

$dW(t) = t^2 dZ(t)$. $\therefore$ Zero drift.

Q13. Let S_t/S_0 follows $\ln$ distⁿ with parameters $(\mu - \frac{1}{2}\sigma^2)t$ and $\sigma^2 t$ such that expected return on ~~a~~ stock is μ & volatility is σ

~~So~~ End of first time step $= S_0 e^{\mu \delta t}$

On the tree price $= q S_0 u + (1-q) S_0 d.$

In order to match the return

$$q S_0 u + (1-q) S_0 d = S_0 e^{\mu \delta t}.$$

$$q = \frac{e^{\mu \delta t} - d}{u-d}$$

Volatity σ of a stock price is defined so that $\sigma\sqrt{\delta t}$ is s.d of return on stock price in a short time period δt

Variance of stock pri return $= qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = \sigma^2 \delta t$

Substituting the value of q in the expression above

$$e^{\mu \delta t}(u+d) - ud - e^{2\mu \delta t} = \sigma^2 \delta t$$

$$\therefore u = e^{\sigma\sqrt{\delta t}} \quad \& \quad d = \frac{1}{u} = e^{-\sigma\sqrt{\delta t}}$$

ii. t=0 t=1 t=2 Payoff @ k=200

t=0	t=1	t=2	Payoff @ k=200
		244.79 221.26	0
	221.26 210.36 0		
		200 206.85	0
200			
		200 193.38	6.62
	180.78 190.15 9.85		
		163.41 180.78	19.22

Value of option at time 1

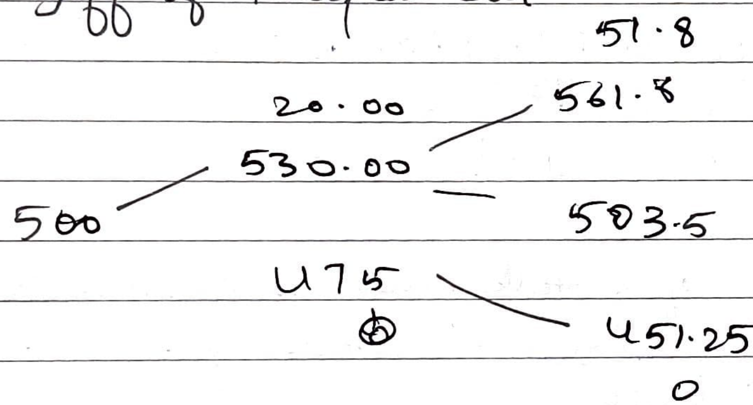
$\left(6.62 \times q + 19.22(1-q)\right) e^{-0.1 \times 1/12} = 12.61$

this is more than payoff at time 1. Hence an American option will not be exercised.

Value of put option at time 0 is

$12.61 \times e^{-0.1 \times 1/12} \times (1-q) = 6.05$

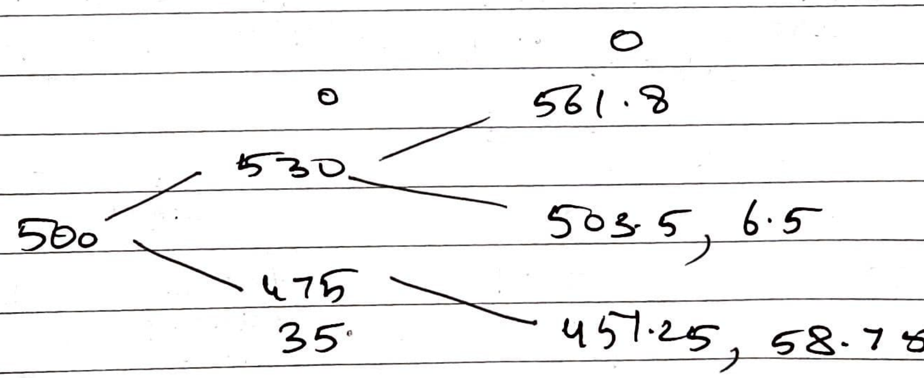
Q14. i. Payoff of European call



Value of the 6 month European call

$= e^{-0.5 \times 0.005} \times 51.8 \times 0.5689^2 = 16.351$

ii.



Value $= e^{-0.05 \times 0.5} \times \left[0.4311^2 \times 58.75 + 2 \times 0.4311 \times 0.5689 \times 6.5\right]$

$= 13.759$.

Value of put + stock $= 500 + 13.759 = 513.759$

Value of call + discounted price $= 16.351 + 510 \times e^{-0.05 \times 0.5}$

$= 513.759$

iii. For the american option, value at final nodes is same ~~for an~~ European option. At earlier nodes, the value of option is greater of value calculated at the node and payoff from early exercise.

We can find that value at node B ~~is~~

$= 6.50 \times e^{-0.25 \times 0.05} \times 0.4311 = 2.767$

Value at node C $= e^{-0.25 \times 0.05} \times (6.50 \times 0.5689 \times 58.75 \times 0.4311) = 28.65$

However, payoff from immediate exercise is 35 at node C. Hence it is advisable to exercise the put option at this point.

Value of american put option at time $t=0$

$e^{-0.05 \times 0.25} \times 35 \times 0.4311 + 2.767 \times 0.5689 = 16.46$

iv. Expected payoff in 3 months time is calculated by using real rate of return of 9%.

$p = 0.6614$ Expected payoff $= 20 \times 0.6614 = 13.228$

~~Unfortunately~~ Risk neutral valuation solves this problem under risk neutral valuation all assets are expected to earn risk free rate.

Q2. $dX_t = \alpha\mu(T-t)\,dt + \sigma\sqrt{(T-t)}\,dZ_t$

Z is a standard Brownian motion, therefore

$dZ_t = 1 \times dZ_t + 0 \times dt$

$G = f(x,t) = e^{m(T-t) - x}$

We know that $f(x,t)$ is a martingale

Given m is a differentiable f^n.

By Ito's lemma

$$dG = df(x,t) = \frac{df}{dt}\,dt + \frac{df}{dx}\,dx_t + \frac{1}{2}\frac{d^2f}{dx^2}(dx_t)^2$$

$$dG = G.\frac{dm}{dt}.dt - G.(\alpha y(T-t)dt + \sigma\sqrt{(T-t)}\,dz_t)$$

$$+ \frac{1}{2}G(\alpha y(T-t)\,dt. + \sigma\sqrt{(T-t)}\,dz_t)^2$$

$$dG = G.\frac{dm}{dt}.dt - G.(\alpha y(T-t))dt - G.(\sigma\sqrt{(T-t)}\,dz_t)$$

$$+ \frac{1}{2}G\sigma^2(T-t)\,dt$$

$$dG = dt\left(G.\frac{dm}{dt} - G(\alpha y(T-t)) + \frac{1}{2}G\sigma^2(T-t)\right)$$

$$- G(\sigma\sqrt{(T-t)}\,dz_t)$$

As we know $f(x,T)$ is a martingale,
This implies it has zero drift.

$$dt\left(G.\frac{dm}{dt} - G(\alpha y(T-t)) + \frac{1}{2}G\sigma^2(T-t)\right) = 0$$

$$dt.G\left(\frac{dm}{dt} - (\alpha y(T-t)) + \frac{1}{2}\sigma^2(T-t)\right) = 0$$

$$\frac{dm}{dt} = -(\alpha y(T-t)) + \frac{1}{2}\sigma^2(T-t))$$

$$\frac{dm}{dt} = \frac{1}{2}\sigma^2(T-t) - (\alpha y(T-t))$$