Assignment 1 (P&S 2)

Q1) $X \sim Poi(\theta)$ $\mu = \theta$

$\sigma^2 = \theta$

$\sigma = \sqrt{\theta}$

$$\frac{X-\theta}{\sqrt{\theta}} \sim N(0,1)$$

$\hat{\theta} = 200$, $n = 1$

$$95\% \ CI = \mu \pm Z_{2.5\%} \ \sigma/\sqrt{n}$$
$$= 200 \pm 1.06\sqrt{200}$$
$$= 200 \pm 27.71858582.$$

$$95\% \ CI = (172.2814142, 227.71858582)$$

Q2.) $X_i = 1, 2, 3, \ldots n.$ (iid)

$E(X_i) = \mu$

$V(X_i) = \sigma^2$

(i) $\bar{X} = \frac{\sum_1^n X_i}{n}$

$$E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right) = \frac{1}{n}\sum E(X_i) = \frac{1}{\cancel{n}} \times \cancel{n} \times \mu = \mu.$$

$$V(\bar{X}) = V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2}\sum V(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

(ii) $\Sigma(s^2) = \sigma^2$

$$s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$$

$$E(s^2) = E\left[\frac{1}{n-1}\Sigma(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n-1} E\left[\Sigma(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n-1} \Sigma E(x_i)^2 - nE(\bar{x}^2)$$

$$= \frac{1}{n-1} \Sigma(\mu^2+\sigma^2) - n(\mu^2+\sigma^2/n)$$

$$\therefore E(\bar{x}^2) = V(\bar{x}) + [E(\bar{x})]^2$$

$$= \frac{\sigma^2}{n} + \mu^2$$

$$= \frac{1}{n-1}(\cancel{n\mu^2} + n\sigma^2 - \cancel{n\mu^2} - \sigma^2)$$

$$= \frac{(n\sigma^2 - \sigma^2)}{n-1}$$

$$= \frac{\sigma^2\cancel{(n-1)}}{\cancel{n-1}} = \sigma^2$$

$$\therefore E(s^2) = \sigma^2$$

(iii) $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

$$V\left[\frac{(n-1)s^2}{\sigma^2}\right] = V(\chi^2_{n-1})$$

$$\therefore \frac{(n-1)^2 V(s^2)}{\sigma^4} = 2(n-1)$$

$$\frac{(n-1)^{\cancel{2}} V(s^2)}{\sigma^4\cancel{(n-1)}} = 2$$

$$\therefore V(s^2) = \frac{2\sigma^4}{(n-1)}$$

(iv) $X_i \sim N(\mu, \sigma^2)$

$\bar{X} \sim N(\mu, \sigma^2/n)$

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$\sigma^2 = 100,\ n = 10$

$\therefore E(s^2) = \sigma^2 = 100$

The values for $E(s^2)$ can be 50 to 150

$P(s<50)$

$$P\left(\frac{(n-1)s^2}{\sigma^2} < \frac{50\times 9}{100}\right)$$

$$P\left(\chi^2_{n-1} < 4.5\right)$$

$$P\left(\chi^2_9 < 4.5\right)$$

$= 0.1245$

$P(s<150)$

$$P\left(\frac{(n-1)s^2}{\sigma^2} < \frac{150\times 9}{100}\right)$$

$$P\left(\chi^2_{n-1} < 13.5\right)$$

$$P\left(\chi^2_9 < 13.5\right)$$

$= 0.8587$

$\therefore P(50 < s^2 < 150) = \cancel{0.1245}\ 0.8587 - 0.1245$

$P(50 < s^2 < 150) = 0.7342$

Q3.)

No. of C	No. of P	E_i	$(O_i - E_i)^2/E_i$
0	86	88.5	0.0706
1	75	68.7	0.5777
2	16	20	0.8
3	2	2.6	
4	1	0.2	

$X \sim Bin(4, P)$

No. of C	O_i	E_i	$(O_i - E_i)^2/E_i$
0	86	33.5	0.0706
1	75	68.7	0.5777
2+	19	22.8	0.6333
			1.2816

$P(X=x) = {}^nC_x (P)^x (1-P)^{n-x}$

$\therefore L = \prod_{i=1}^{n} {}^nC_x (P)^x (1-P)^{n-x}$

$= P(x=0)^{86} \, P(X=1)^{75} \cdot P(X=2)^{16} \cdot P(X=3)^2 \cdot P(X=4)^1$

$= \left[{}^4C_0 (P)^0 (1-P)^4\right]^{86} \cdot \left[{}^4C_1 (P)^1 (1-P)^3\right]^{75} \cdot \left[{}^4C_2 (P)^2 (1-P)^2\right]^{19} \cdot \left[{}^4C_3 (P)^3 (1-P)^1\right]^2$
$\left[{}^4C_4 (P)^4 (1-P)^0\right]^1$

$\ln L = 86 \times 4 \log(1-P) + 75 \ln P + 225 \ln(1-P) + 32 \ln P + 32 \ln(1-P)$
$+ 6 \ln P + 2 \ln(1-P) + 4 \ln P$

$\frac{d \ln L}{dP} = 86 \frac{-344}{1-P} + \frac{75}{P} - \frac{225}{1-P} + \frac{32}{P} - \frac{32}{1-P} + \frac{6}{P} - \frac{2}{1-P} + \frac{4}{P}$

$\therefore -603p + 117 - 117p = 0$

$\therefore 117 = 720p$

$\hat{p} = 0.1625$

$\therefore P(X=0) = (1-p)^4 = 0.49197$

$P(X=1) = 4p(1-p)^3 = 0.38183$

$P(X=2) = 6p^2(1-p)^2 = 0.11113$

$P(X=3) = 4p^3(1-p) = 0.01437$

$P(X=4) = p^4 = 0.00080$

$\therefore \Sigma Z^2 = 1.2816$

$\chi^2_{1, 95\%} = 3.841$

$df = n-1-\text{no. of p} - \text{no. of c.}$

$= 5-1-1-2$

$= 1$

$\therefore Z^2_{cal} < Z^2_{tab}$

$\therefore$ Do not reject

The above data does not follow binomial dist^n

Q5.) $X \sim U(a,b)$

(i) $\therefore \bar{x} = a+b/2$

$s^2 = (b-a)^2/12$

$\therefore s = b-a/\sqrt{12}$

$2\sqrt{3}\,s = b-a$

$\hat{b} = \hat{a} + 2\sqrt{3}s$ — (i)

$\therefore \bar{x} = (a+b)/2$

$\bar{x} = (a+a+2\sqrt{3}s)/2$ — (from (i))

$\bar{x} = (2a+2\sqrt{3}s)/2$

$\bar{x} = \hat{a} + \sqrt{3}s$

$\hat{a} = \bar{x} - \sqrt{3}s$

(ii) $\hat{a} = \bar{x} - \sqrt{3}s$

$\hat{b} = \hat{a} + 2\sqrt{3}s$

$\hat{a} + 2\sqrt{3}s = \bar{x} - \sqrt{3}s + 2\sqrt{3}s$

$\therefore \hat{b} = \bar{x} + \sqrt{3}s$

(iii) Sample $\longrightarrow 1, 2, 4, 20$

$\bar{x} = \frac{30}{5} = 6$

$S = 7.9057$

$\hat{a} = 6 - \sqrt{3}(7.9057)$

$= -7.6930$

$\hat{b} = 6 + \sqrt{3}(7.9057)$

$= 19.6930$

$\therefore$ The estimate values of $\hat{a}$ & $\hat{b}$ are not accurate.

(iv) $x \sim U(0, b)$

$f(x) = \frac{1}{b-a} = \frac{1}{b}$

$L = f(x_1) \cdot f(x_2) \cdot f(x_3) \ldots f(x_n)$

$= \frac{1}{b} \cdot \frac{1}{b} \cdots \frac{1}{b}$

$L = \frac{1}{b^n}$

$\ln L = -n \ln b$

$\frac{d \ln L}{db} = -\frac{n}{b}$

$$\frac{d \ln L}{db} = \frac{-n}{b} = 0, \; b \to \infty$$

$$\frac{d^2 \ln L}{db^2} = \frac{+n}{b^2} > 0, \text{ min at } b \to \infty$$

~~Q6.) $H_0 : p = 0.1$~~
~~$H_1 : p > 0.1$~~

Q7.)

(i) Public | Private
$\bar{x}_1 = 30$ | $\bar{x}_2 = 35.06$
$S_1^2 = 64.31$ | $S_2^2 = 67.42$

(ii) $H_0 \Rightarrow \mu_1 \Rightarrow \mu_2$, $H_1 \Rightarrow \mu_1 \neq \mu_2$

$$\frac{(30 - 35.06) - 0}{\sqrt{65.8574} + \sqrt{2/9}} = 1.32$$

$t_{n_1+n_2-2} = t_{16}$ at 5% LoS $= (-2.12, 2.12)$

Donot ~~reg~~ reject H_0.
The population means are equal at 5% LoS.

Q8.)

(i) $\hat{\theta}$ is said to be unbaised when $E(\hat{\theta}) = \theta$

(ii) Measure of bias is given by $E(\hat{\theta}) - \theta$

(iii) M. SE of $\hat{\theta} = [E(\hat{\theta}) - \theta]^2$

(iv) $\tilde{\theta}$ is a more efficient estimate because it will have a lower varlance, while $\hat{\theta}$ having no bias has a bigger variance.

(v) An estimate is said to be consistent, when MSE converges to zero as $n \to \infty$

(vi) Method of Moments (MOM).
Maximum Likelihood Generator (MLE)

Q a.)

(i) $t_k = \dfrac{Z}{\sqrt{\chi_k^2 / K}}$

$Z \sim N(0,1)$

$\chi_k^2 \Rightarrow$ chi-square distribution with k ~~d of~~ degrees of freedom

$k =$ degree of freedom.

(ii) $E(t_k) = 0 \qquad V(t_k) = \dfrac{k}{k-2} \qquad$ for $k > 2$

(iii) $n = 10$

$\bar{x} = 50$

$s^2 = 48.667$

(a) $\dfrac{\bar{x} - \mu}{S/\sqrt{n}} \sim t_{n-1}, \quad t_9 = 3.25$

CI for 99% $= 50 \pm 3.25 \sqrt{\dfrac{48.667}{10}}$

$= (42.83, 57.17)$

Q11.)

Q11.)

$\Sigma x = 213200$

$\Sigma x^2 = 4919860000$

$\bar{x} = 17767$

$S = 10144$

$\Sigma x_1 = 213200 - 11000 - 48000 + 27500 + 31500$

$\Sigma x_1 = 213200$

$$\bar{x}_1 = \frac{\Sigma x_1}{n} = \frac{213200}{12} = 17767$$

$\Sigma x_1^2 = 4919860000 - (11000)^2 - (48000)^2 + (27500)^2 + (31500)^2$

$\therefore \Sigma x_1^2 = 4243360000$

$$S_1 = \frac{\sqrt{4243360000 - 12(17767)^2}}{11}$$

$$S_1 = \frac{\sqrt{4243360000 - 12 \times 315666289}}{11}$$

$\cancel{= 4}$

$$= \frac{\sqrt{4243360000 - 3787995468}}{11}$$

$$= \frac{\sqrt{455364532}}{11}$$

$$= \frac{21339.2720588}{11}$$

$S_1 = 1939.93382353$