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Semester 4
Assignment Number 2

Statistical and Risk Modelling 2

Q 1] (i)
Ans. The generator matrix can be given as,

$$A = \begin{array}{c c} & \begin{array}{cc} B & O \end{array} \\ \begin{array}{c} B \\ O \end{array} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

(ii)
Ans The holding times are exponentially distributed with parameter λ in state B and μ in state O.

(iii)
Ans.
$$\frac{d}{dt} {}_tp_0^{BB} = -\lambda {}_tp_0^{BB} + \mu \cdot {}_tp_0^{BO}$$

$$\frac{d}{dt} {}_tp_0^{BO} = \lambda {}_tp_0^{BB} - \mu {}_tp_0$$

(iv)
Ans. Considering, we have a two state model,

$${}_tp_0^{BB} + {}_tp_0^{BO} = 1$$

Substituting,

$$\frac{d}{dt} {}_tp_0^{BB} = -\lambda {}_tp_0^{BB} + \mu(1 - {}_tp_0^{BB})$$

$$\therefore \frac{d}{dt}\left[e^{(\lambda+\mu)t} \, {}_tP_0^{BB}\right] = \mu e^{(\lambda+\mu)t}$$

$$\therefore e^{(\lambda+\mu)t} \cdot {}_tP_0^{BB} = \frac{\mu}{\lambda+\mu} e^{(\lambda+\mu)t} + \text{constant}$$

$$\therefore \text{constant} = \frac{\lambda}{\mu+\lambda}$$

Thus,

$${}_tP_0^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

Q.2] (i)

Ans: The transition graph from the process can be drawn using the given generator matrix

(ii)

Ans:

$$\frac{d}{dt} P_{AA}(t) = -0.3 P_{AA}(t) + 0.1 P_{AB}(t) + 0.3 P_{AC}(t)$$

$$\frac{d}{dt} P_{AB}(t) = 0.2 P_{AA}(t) - 0.5 P_{AB}(t) + 0.1 P_{AC}(t)$$

$$\frac{d}{dt} P_{AC}(t) = 0.1 P_{AA}(t) + 0.4 P_{AB}(t) - 0.4 P_{AC}(t)$$

(iii)

Ans: To stay in state A, the equation reduces to,

$$\frac{d}{dt} P_{\overline{AA}}(t) = -0.3 P_{AA}(t).$$

$$p_{\overline{AA}}(t) = e^{-0.3t}$$

Thus, for $t = 2$

$$p_{\overline{AA}}(2) = e^{-0.3\times 2}$$

$$p_{\overline{AA}}(2) = 0.5488$$

(iv)

Ans. The only paths under which the third jump is into state C are BAC, CAC and CBC.

Thus, the probabilities of each path are,

$$BAC = \frac{2}{3}\times\frac{1}{5}\times\frac{1}{3} = \frac{2}{45}$$

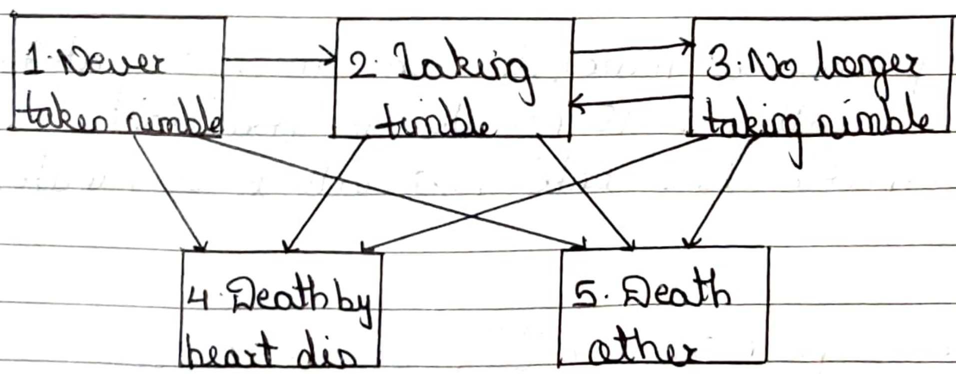
$$CAC = \frac{1}{3}\times\frac{3}{4}\times\frac{1}{3} = \frac{1}{12}$$

$$CBC = \frac{1}{3}\times\frac{1}{4}\times\frac{4}{5} = \frac{1}{15}$$

Thus, total probability $= 0.194$

Q.3] (i)

Ans.



(ii)

Ans. Using the Markov assumption,

The Chapman Kolmogorov equation is,

$$ {}_{dt+t}p_x^{34} = {}_tp_x^{31}\,{}_{dt}p_{x+t}^{14} + {}_tp_x^{32}\,{}_{dt}p_{x+t}^{24} + {}_tp_x^{33}\,{}_{dt}p_{x+t}^{34} + {}_tp_x^{34}\,{}_{dt}p_{x+t}^{44} + {}_tp_x^{35}\,{}_{dt}p_{x+t}^{54}$$

Since, $_{dt}p_{x+t}^{54} - {}_{t}p_{x}^{31} = 0$

$$_{dt}p_{x+t}^{34} = {}_{t}p_{x}^{32} \cdot {}_{dt}p_{x+t}^{24} + {}_{t}p_{x}^{33} \cdot {}_{dt}p_{x+t}^{34} + {}_{t}p_{x}^{34} \cdot {}_{dt}p_{x+t}^{44}$$

Given that, $_{dt}p_{x+t}^{44} = 1$

And, assuming that, for small dt,

$$_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij}\, dt + o(dt) \quad , i \neq j$$

Thus,

$$_{dt+t}p_{x}^{34} - {}_{t}p_{x}^{34} = {}_{t}p_{x}^{32} \cdot \mu_{x+t}^{24}\, dt + {}_{t}p_{x}^{33} \cdot \mu_{x+t}^{34}\, dt + o(dt)$$

Q.4] (i)
Ans: The mean is equal to the parameter, so there are 3 calls per hour.

(ii)
Ans: The process is memoryless so the fact that Fred has not had a call for 15 minutes is irrelevant.
∴ Expected time until next call is 20 minutes.

(iii)
Ans: Since, $P_0(0.5) = \dfrac{e^{-1.5}(1.5)^0}{0!} = 0.2231$

(iv)
Ans: The expected time that Fred is on the phone is the expected number of calls times the expected length of a call. Per hour that is 3 calls times 7 minutes = 21 minutes.
So, the probability that the phone is engaged = 21/60 = 0.3

Q.5] (i)

Ans The Chapman Kolmogorov equation is,

$$p_{HH}(x, t+dt) = p_{HH}(x,t) \cdot p_{HH}(t, t+dt) + p_{HS}(x,t) \cdot p_{SH}(t, t+dt) + p_{HD}(x,t) \cdot p_{DH}(t, t+dt)$$

$$\therefore p_{HH}(x, t+dt) = p_{HH}(x,t) p_{HH}(t, t+dt) + p_{HS}(x,t) \cdot p_{SH}(t, t+dt)$$

Assuming that, for small dt,

$$p_{ij}(t, t+dt) = \lambda_{ij}(t)\, dt + o(dt) \quad , i \neq j$$

$$p_{ii}(t, t+dt) = 1 + \lambda_{ii}(t)\, dt + o(dt)$$

$$\therefore \frac{d}{dt} p_{HH}(x,t) = p_{HH}(x,t) \cdot (-\sigma(t) - \mu(t)) + p_{HS}(x,t)\, \rho(t, C_t)$$

(ii)

Ans

$$\frac{d}{dt} p_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t)) \cdot p_{\overline{HH}}(t)$$

$$\therefore \frac{d}{dt} \ln p_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))$$

Integrating on both the sides,

$$\left[\ln p_{\overline{HH}}(0,t)\right]_0^t = \int_0^t -[\sigma(s) + \mu(s)]\, ds$$

$$\therefore p_{\overline{HH}}(0,t) = e^{-\int_0^t (\sigma(s) + \mu(s))\, ds}$$

Q.6]

Ans The similarities and differences between Markov chain, markov jump chain and markov jump process are as follows:

1 All the mentioned processes have a discrete state space

2 Markov chain and markov jump chain operates in discrete time but a markov jump process operates in continuous time.

3 All the mentioned processes satisfy the markov property.

4 Markov jump chain obeys the Markov property and behaves as a markov chain except when the jump encounters an absorbing state.

5 Markov jump chain and markov chain are expressed in probabilities whereas markov jump process is expressed in terms of rates.

Q.7] (i)

Ans The maximum likelihood estimates of the transition intensity from state i to state j is the number of transitions from state i to state j divided by the total waiting time in state i.

(ii)

Ans

$$\frac{d}{dt} P_{AA}(s,t) = -P_{AA}(s,t) \cdot \mu$$

$$\frac{d}{dt} P_{AT}(s,t) = P_{AA}(s,t) \cdot \mu$$

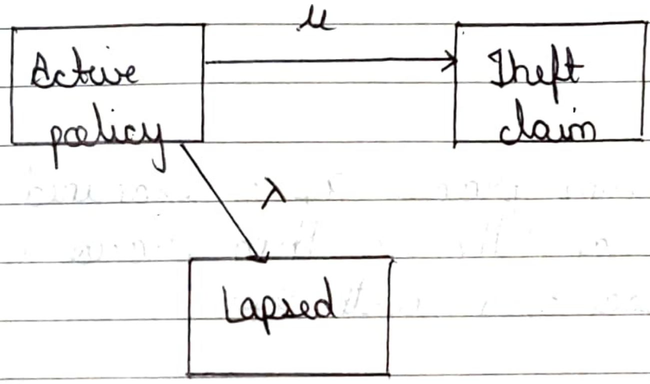
$$\therefore \frac{d}{dt} P_{AT}(s,t) = (1 - P_{AT}(s,t))\, \mu$$

Using an integrating factor,

$$p_{AT}(t) = 1 - e^{-\mu t}$$

$\therefore$ Expected cost is $c\left(1 - e^{-\mu t}\right)$

(iii)
Ans:



(iv)
Ans: We now have,

$$\frac{d}{dt} p_{AA}(t) = -p_{AA}(t)(\mu+\lambda)$$

So, $p_{AA}(t) = e^{-(\mu+\lambda)t}$

So, claim becomes $\dfrac{\mu}{\mu+\lambda} c\left(1 - e^{-(\mu+\lambda)T}\right)$

Q.8] (i)
Ans: $\{0, 1, 2, 3, 4, \dots\}$

(ii)
Ans:

$$0 \xleftarrow[\mu]{} 1 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 2 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 3 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 4 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \cdots$$

(iii)
Ans

$$\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdot \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) & \\ \cdot & & & & & \cdot \end{bmatrix} \end{array}$$

(iv)
Ans If a Markov jump process X_t is examined only at the time of transition, the resulting process is called the jump chain associated with X_t.

(v)
Ans

$$\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \text{etc} \\ \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 & \\ 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & \\ 0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & \\ 0 & 0 & 0 & \mu/(\mu+\lambda) & 0 & \\ \text{etc} & & & & & \end{bmatrix} \end{array}$$

(vi)
Ans $\left(\frac{\mu}{\mu+\lambda}\right)^3$

Q.9] (i)
Ans A generator matrix is a matrix whose entries signify transition rates from one state to every other in a Markov jump process.

$$\mu_{ii} = -\sum_{i \neq j} \mu_{ij}$$

(ii)
Ans. Required state space is $\{0, 1, 2\}$, since the 2 policies in force can still be in force at a future time, either of the policies can be claimed or 'lapsed' or both can expire at a future time

(iii)
Ans.

$$\begin{tikzcd} \boxed{0} & \boxed{1} \arrow[l, "\mu_{10}"'] & \boxed{2} \arrow[l, "\mu_{21}"'] \end{tikzcd}$$

Q.10] (i)
Ans

$$\frac{d}{dt} p_{\overline{AA}}(t) = -2t \times p_{\overline{AA}}(t)$$

$$\therefore \ln p_{\overline{AA}}(s) = -s^2 + \text{constant}$$

We know, $p_{\overline{AA}}(0) = 1$

$$\therefore p_{\overline{AA}}(s) = e^{-s^2}$$

(ii)
Ans. P (in first visit to B at time T in state A at t = 0)

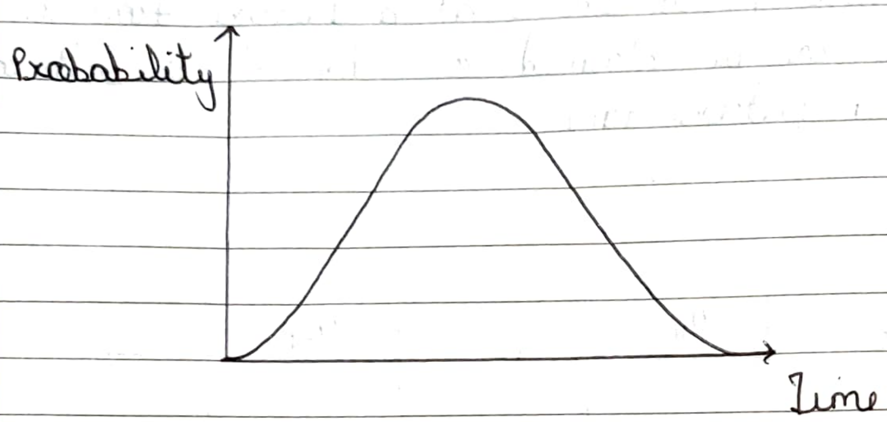
$$= \int_0^T p_{\overline{AA}}(s) \times 2s \times p_{\overline{BB}}(s, T)\, ds$$

$$= \int_0^T e^{-s^2} \times 2s \times e^{-T^2 + s^2}\, ds$$

$$= e^{-T^2} \times T^2$$

(iii)

Ans: The sketch should be shaped like :



Q.11] (i)

Ans: Let N_t denote the number of claims up to time t. Since the poisson process has stationary increments, we may take $t = 0$, so that the required conditional distribution is

$$P(T_0 \leq y \mid N_s = 1) = \frac{P(T_0 \leq y, N_s = 1)}{P(N_s = 1)}$$

$$= \frac{(\lambda y e^{-\lambda y}) e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}}$$

$$= \frac{y}{s}$$

$$= \text{cdf of } U[0, s]$$

(ii)

Ans: Since, holding times are independent, each having an exponential distribution, their joint density is,

$$\lambda^n e^{-\lambda(t_1 + t_2 + \dots + t_n)} 1_{\{t_1, t_2, \dots, t_n > 0\}}$$

(iii)

Ans. We got,

$$P(N_s = k \mid N_t = n) = \frac{P(N_s = k,\ N_t - N_s = n-k)}{P(N_t = n)}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^k}{k!} \times \dfrac{e^{-\lambda(t-s)} \lambda^{n-k} (t-s)^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!} \frac{s^k (t-s)^{n-k}}{t^k t^{n-k}}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

= Binomial with parameters n and s/t.