1.(i) $S = \{B, O\}$; $A = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$

ii) $T_i \sim Exp[\lambda_i]$.

iii) $$\frac{d}{dt}\, {}_tP_s = \lambda\, {}_tP_s^{BB} + \mu\, {}_tP_s^{BO}$$

$$\frac{d}{dt}\, {}_tP_s^{BO} = \lambda\, {}_tP_s^{BB} - \mu\, {}_tP_s^{BO}$$

iv) $$\frac{d}{dt}\, {}_tP_s^{BB} = -\lambda\, {}_tP_s^{BB} + \mu(1 - {}_tP_s^{BB})$$

$$\frac{d}{dt} \exp[(\lambda+\mu)t)\, {}_tP_s^{BB}] = \mu \exp[(\lambda+\mu)t].$$

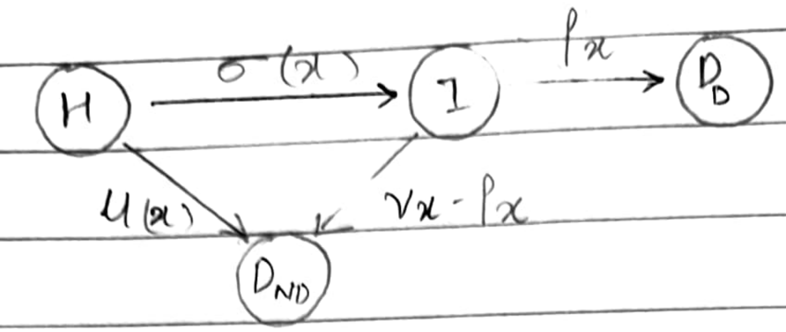
$$\Rightarrow \exp[(\lambda+\mu)t]\, {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} \exp[(\lambda+\mu)t] + c$$

At $t=0$, $c = \frac{\lambda}{\mu+\lambda}$

$$\therefore {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot e^{-(\lambda+\mu)t}$$

2)

i)



Total mortality rate of infected state $= \nu(x)$

Mortality rate from disease $= \nu(x)\,\rho(x)$

~~ii)~~ $\cancel{P(t) = P_{HH}(t)\; P_{HI}(t)\; P}$

ii) Matrix form of Kolmogorov's forward equation

$$\frac{d}{dt} P(s,t) = P(s,t)\cdot A(t)$$

$$A(t) = \begin{bmatrix} -(\sigma_t+\mu_t) & \sigma_t & 0 & \mu_t \\ 0 & -\nu_t & \rho_t & \nu_t-\rho_t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

iii)

a) Given that a person cannot go back to being healthy,

$P_{HH}(x, x+t) = P_{\overline{HH}}(x, x+t)$

$\therefore P_{HH}(x, x+t) = \exp\left[-\int_x^{x+t} \sigma_u + \mu_u \, du\right]$

Date
Page

b) $P_{HI}(x, x+t)$

$\cancel{P_{HH}(x, x+t) = P_{HH}(x, x+t-w)}$

$$P_{HI}(x, x+t) = P_{HH}(x, x+t-w)\,\sigma(w)\,P_{\bar{I}I}(w)$$

Allowing last jump to occur at any time.

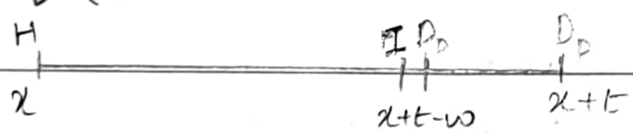
$$P_{HI}(x, x+t) = \int_{x}^{x+t} P_{\bar{H}H}(x, x+t-w)\,\sigma_{x+t-w}\,P_{\bar{I}I}(x+t-w, x+t)\,dw.$$

where,

$$P_{\bar{H}H} = \exp\left[-\int^{x+t} \sigma_u + \mu_u \, du\right]$$

$$P_{\bar{I}I} = \exp\left[-\int_{x}^{x+t} \rho_u + \nu_u - \rho_u \, du\right] = \exp\left[-\int_{x}^{x+t} \nu_u \, du\right].$$

c) $P_{HD_D}(x, x+t)$



$$P_{HD_D}(\cancel{x, x+t}) = \cancel{\int_{x}^{x+t} P_{\bar{H}H}(x, x+t-w)\,\mu_{x+t-w}\,P_{\bar{D}D}\,dw}.$$

$$P_{HD_D}(x, x+t) = \int_{x}^{x+t} P_{HI}(x, x+t-w)\,\rho_{x+t-w}\,dw.$$

Date
Page

3) $\Phi_{HH} = \mu$, $\Phi_{HL} = \rho$.

i) $$(H) \underset{\rho}{\overset{\mu}{\rightleftarrows}} (L)$$

$$A = \begin{bmatrix} -\mu & \mu \\ \rho & -\rho \end{bmatrix}$$

ii) Let T_j be the holding time RV for state j.
$T_j \sim Exp(\lambda_j)$.
$\therefore T_H \sim Exp(\mu)$
$T_L \sim Exp(\rho)$.

iii) Firstly, we will need the number of jumps from high to low (low to high). And time spent in each state.
Then Likelihood function is:-
$$L = \mu^{N_{HL}} \rho^{N_{LH}} \exp[-\text{Total time in H}\cdot\mu] \cdot \exp[-\text{Total time in Low} \cdot \rho]$$

Then μ and ρ can be estimated using standard MLE approach.

iv) $\frac{d}{dt} P(s,t) = P(s,t) \cdot A(t)$. Let $T = s+t$.

$$\Rightarrow \begin{bmatrix} P_{HH}(S,T) & P_{HL}(S,T) \\ P_{LH}(S,T) & P_{LL}(S,T) \end{bmatrix} \begin{bmatrix} -\mu & \mu \\ \rho & -\rho \end{bmatrix}$$

$$\therefore \frac{d}{dt} {}_sP_t^{LH} = \frac{d}{dt} P_{LH}(S,T) = \mu P_{LH}(S,S+t) - \rho P_{LH}(S,S+t)$$
$$= \mu\, {}_sP_t^{LH} - \rho\, {}_sP_t^{LL}.$$

Date
Page

$$\frac{d}{dt}\,{}_sP_t^{LH} = \rho \cdot {}_sP_t^{LL} - \mu\, {}_sP_t^{LH}$$

v)

vi) ${}_tP_0^{LL} = P_{LL}(0,t)$

$$\frac{d}{dt} P_{LL}(0,t) = \mu P_{LH}(0,t) - \rho P_{LL}(0,t)$$

$$\Rightarrow \frac{d}{dt} P_{LL}(t) + \rho P_{LL}(0,t) = \mu P_{LH}(t)$$

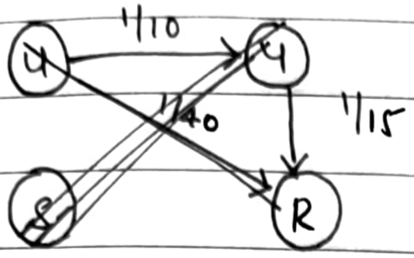
$P = \rho,\ Q = \mu P_{LH}(t)$.

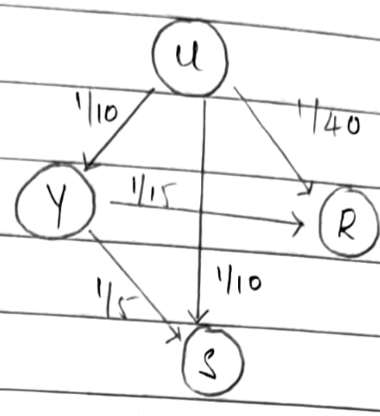
$IF = \exp\left[\int P\,dt\right] = \exp\left[\rho t + c\right]$.

$$\therefore P_{LL}(t)\, e^{\rho t} = \int \mu P_{LH}(t)\, e^{\rho t}\, dt + c.$$

5. Uncautioned : U.
Yellow : Y
Red : R.
Substituted : S.

i)





ii) $A = \begin{bmatrix} -9/40 & 1/10 & 1/40 & 1/10 \\ 0 & -4/15 & 1/15 & 1/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$P_{UU} \quad P_{UY} \quad P_{UR} \quad P_{US} \qquad -\frac{9}{40} \quad 1/10 \quad 1/40$

$\frac{P_{UU}}{40} + \frac{P_{UY}}{15} \qquad 1/15$

Forward Equations are given by:

$$\frac{d}{dt} P(t) = P(t) \cdot A.$$

iii) $\frac{d}{dt} P_{\overline{UU}}(1.5) = -\frac{9}{40} P_{\overline{UU}}(1.5)$

$P_{\overline{UU}}(1.5) = e^{-9/40 \cdot 1.5} = \underline{\underline{0.71355}}.$

$\frac{d}{dt} P_{\overline{UU}}(t) = \frac{-9}{40} P_{\overline{UU}}(t)$

$= \frac{-9}{40}\, e^{-9/40 t}$

$\int_0^n \frac{d}{dt} P_{\overline{UU}}(t) = \int_0^n \frac{-9}{40} e^{-9/40 t} dt = [P_{\overline{UU}}(t)]_0^n = \cancel{\frac{-9}{40}} \left[\frac{e^{-9/40 t}}{\cancel{-9/40}}\right]_0^n$

$P_{\bar{u}\bar{u}}(t) \cancel{-1} = e^{-\frac{9t}{40}} \cancel{-1}$

$\therefore P_{\bar{u}\bar{u}}(1.5) = \underline{\underline{0.71355}}$

iv) ~~$P_{uR}(t)$~~ $\cancel{\frac{d}{dt} P_{uR}(t) = \frac{P_{uu}}{40} + \frac{P_{uu}}{15}}$

$$P_{uR}(t) = \int_0^{1.5} P_{\bar{u}\bar{u}}(s) \mu_{uR} \, ds.$$

$$= \int_0^{1.5} e^{-\frac{9s}{40}} \cdot \frac{1}{40} \, ds.$$

$$= \frac{1}{40} \left[\frac{e^{-9s/40}}{\frac{-9}{40}} \right]_0^{1.5}$$

$$= -\frac{1}{9} \left[e^{-9s/40} \right]_0^{1.5}$$

$P_{uR}(t) = \underline{\underline{0.03183}}.$

v) a) $P_{uR}(\infty) = -\frac{1}{9} \left[e^{-9s/40} \right]_0^{\infty}$

$$= \underline{\underline{\frac{1}{9}}}$$

6) i) A: Alive

I: Dead from illness.

D: Dead from sacrifice

Z: Zombie

$$\begin{tikzcd}
A \arrow[r, "\mu_{AI}"] \arrow[d, "\mu_{AD}"'] & I \arrow[d, "\mu_{IZ}"] \\
D \arrow[r, "\mu_{DZ}"'] & Z
\end{tikzcd}$$

ii) Let t_i denote time in state i.

n_{ij} denote transition from state i to j.

$$\therefore L = \mu_{AI}^{n_{AI}} e^{-t_A\mu_A} \mu_{IZ}^{n_{IZ}} e^{-t_I\mu_I} \mu_{AD} e^{-t_A\mu_A} \mu_{DZ}^{n_{DZ}} e^{-t_D\mu_D}$$

$$= \mu_{AI}^{n_{AI}} e^{-2t_A\mu_A} \mu_{IZ}^{n_{IZ}} e^{-t_I\mu_I} \mu_{AD}^{n_{AD}} \mu_{DZ}^{n_{DZ}} e^{-t_D\mu_D}.$$

where, μ_i is total force of transition out of state i.

iii) $$\ln L = n_{AI}\ln\mu_{AI} - 2t_A\mu_A + n_{IZ}\ln\mu_{IZ} + \dots - t_D\mu_D$$

$$= n_{AI}\ln\mu_{AI} - 2t_A\mu_{AI} - 2t_A\mu_{AD} + \dots - t_D\mu_D$$

$$\frac{d}{d\mu_{AI}}\ln L = \frac{n_{AI}}{\mu_{AI}} - t_a = 0.$$

$$\Rightarrow \hat{\mu}_{AI} = \frac{n_{AI}}{t_A}.$$

$$\frac{d^2}{d\mu_{AI}^2}\ln L = -\frac{n_{AI}}{\mu_{AI}^2} < 0 \quad \therefore \text{Max}.$$

iv) $y_1 = 46,567$, $y_2 = 46,577$

$n_A = 3,189$. $n_A = 2,811$

$n_{AI} = 3,690$

$n_{AD} = 2,310$.

$$\hat{\mu}_{AI} = \frac{n_{AI}}{t_A} = \frac{3,690}{t_A}$$

$$\hat{\mu}_{AD} = \frac{n_{AD}}{t_A} = \frac{2,310}{t_A}$$

$\Rightarrow n_{AI}$ (yearly) $= 369$.

n_{AD} (yearly) $= 231$.

Assuming population varies linearly b/w census dates.

$t_A = \frac{1}{2}(3690 + 2310) = \underline{\underline{3,000}}$

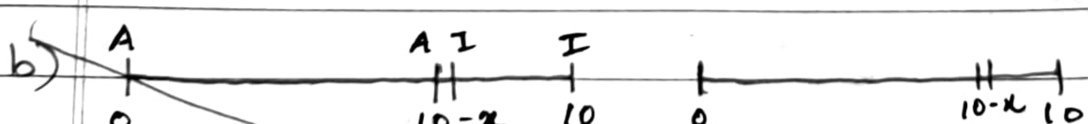
$\therefore \hat{\mu}_{AI} = 0.123$

$\hat{\mu}_{AD} = 0.077$.

v) a) $P_{\bar{A}\bar{A}}(t) = e^{-\lambda_A \cdot t} = e^{-0.2t}$

$\therefore P_{\bar{A}\bar{A}}(10) = \underline{\underline{0.1353}}$.

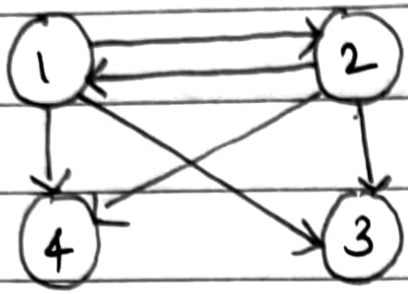
b)



Required Prob $= P_{AI}(10) + P_{AD}(10) - P_{DD}(10) - P_{I2}(10)$

$$= \int_0^{10} P_{\bar{A}\bar{A}}(10)\,\mu_{AI}\,P_{\bar{I}\bar{I}}(10)\,dx + \int_0^{10} P_{\bar{A}\bar{A}}\,\mu_{AD}\,P_{\bar{D}\bar{D}}\,dx - \int_0^{10} P_{\bar{D}\bar{D}}\,\mu_{D2}\,dx - \int_0^{10} P_{\bar{I}\bar{I}}\,\mu_{I2}\,dx$$

7.



i) $L = \mu_{12}^{n_{12}} \cdot \mu_{23}^{n_{23}} \mu_{13}^{n_{13}} \mu_{14}^{n_{14}} \mu_{24}^{n_{24}} \mu_{21}^{n_{21}} \cdot e^{-t_1(\mu_{12}+\mu_{14}+\mu_{13})} \exp[-t_2(\mu_{21}+\mu_{23}+\mu_{24}$

ii) $\ln L = n_{12} \ln \mu_{12} + n_{23} \ln \mu_{23} + n_{13} \ln \mu_{13} + \ln \mu_{14} \cdot n_{14} +$
$n_{24} \ln \mu_{24} + n_{21} \ln \mu_{21} - t_1(\mu_{12} + \mu_{14} + \mu_{13}) -$
$t_2(\mu_{21} + \mu_{23} + \mu_{24})$

$\mu = \mu_{13} + \mu_{14}.$

$$\frac{\partial}{\partial \mu_{13}} \ln L = \frac{n_{13}}{\mu_{13}} - t_1 = 0.$$

$$\Rightarrow \hat{\mu}_{13} = \frac{n_{13}}{t_1} \quad \Rightarrow \tilde{\mu}_{13} = \frac{N_{13}}{t_1}$$

Similarly, $\tilde{\mu}_{14} = \dfrac{N_{14}}{t_1}$

$\Rightarrow \tilde{\mu}_{13} =$

$$\Rightarrow \tilde{\mu}_x = \frac{N_{13}}{t_1} + \frac{N_{14}}{t_1} = \underline{\underline{\frac{N_{13}+N_{14}}{t_1}}}$$

iii) $t_1 = 14,392$.

$t_2 = 18,109$

$n_{13} = 178$.

$n_{23} = 190$

$n_{14} = 89$

$n_{24} = 53$.

μ_1 : Rate of mortality from ~~tran~~ obese

μ_2 : Rate " " " not obese.

$\tilde{\mu}_1 \sim N(\mu_1, CRLB_1)$.

$\tilde{\mu}_2 \sim N(\mu_2, CRLB_2)$.

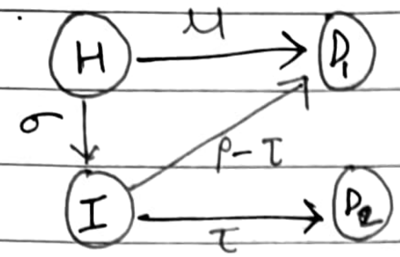
$H_0: \mu_1 > \mu_2$ vs $H_1: \mu_1 \overset{<}{\cancel{>}} \mu_2$.

$\mu_1 - \mu_2 > 0$ vs $H_1: \mu_1 - \mu_2 < 0$.

$\mu_1 - \mu_2 \sim N(0, \sigma_1 + \sigma_2)$.

$$\sigma_1 = \frac{-1}{E\left(\frac{d^2}{d\mu_1^2} \ln L\right)} =$$

3) i)



H: Healthy.
D_1: Dead (other cause)
D_2: Dead (CG)
I: Infected.

ii) $A =$
$$\begin{array}{c|cccc} & H & I & D_1 & D_2 \\ \hline H & -(\sigma+\mu) & \sigma & \mu & 0 \\ I & 0 & -\rho & (\rho-\tau) & \tau \\ D_1 & 0 & 0 & 0 & 0 \\ D_2 & 0 & 0 & 0 & 0 \end{array}$$

Forward Equations:-
$$\frac{\partial}{\partial t} P(t) = P(t) \cdot A.$$

iii) $L = \exp[-1200(\sigma+\mu)]\, e^{-600\rho}\, \mu^{10}\, \tau^{30}$

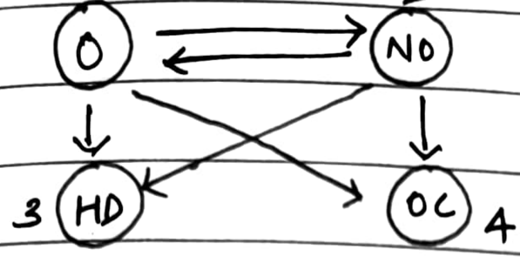
iv) $\ln L = -1200(\sigma+\mu) - 600\rho + 10\ln\mu + 30\ln\tau$ $(\tau+x)$

$$\frac{\partial}{\partial \tau}\ln L = \frac{30}{\tau} - 600 = 0.$$

v) $\Rightarrow \hat{\tau} = \frac{30}{600} = \underline{\underline{0.05}}$

iv) $\therefore \hat{\tau} = \frac{N_{ID_2}}{T_I}$ where, N_{ID_2}: No. of deaths from CG.
T_I: Total time spent infected.

9 2)



i) Let μ_{ij} denote transition intensity from state i to j.
Let n_{ij} denote no. of transitions from state i to j.
Let t_i denote total waiting time in state i.

$$\therefore L = \mu_{12}^{n_{12}} \cdot \mu_{21}^{n_{21}} \cdot \mu_{13}^{n_{13}} \cdot \mu_{23}^{n_{23}} \, \mu_{14}^{n_{14}} \, \mu_{24}^{n_{24}} \exp[-(\mu_{12}+\mu_{13}+\mu_{14})t_1]$$
$$\cdot \exp[-(\mu_{21}+\mu_{23}+\mu_{24})t_2]$$

ii) We need, $\hat{\mu}_{13}$

$$\cancel{\ln L = n_{12} \ln \mu_{12} + \sum_{i=1}^{4}}$$

$$\ln L = n_{12} \ln \mu_{12} + n_{21} \ln \mu_{21} + n_{13} \ln \mu_{13} + n_{23} \ln \mu_{23} + n_{14} \ln \mu_{14} + n_{24} \ln \mu_{24} - \mu_{12} t_1 - \mu_{13} t_1 - \mu_{14} t_1 - \mu_{21} t_2 - \mu_{23} t_2 - \mu_{24} t_2.$$

$$\frac{d}{d\mu_{13}} \ln L = \frac{n_{13}}{\mu_{13}} - t_1 = 0.$$

$$\rightarrow \hat{\mu}_{13} = \frac{n_{13}}{t_1}$$

and, $\tilde{\mu} = \frac{N_{13}}{T_1}$ Estimator.

iii) $\cancel{T}\ t_1 = 14,392$

$t_2 = 18,109$

$n_{13} = 178.$

$n_{23} = 190$

$n_{14} = 89$

$n_{24} = 53.$

$H_0 : \mu_{13} = \mu_{23} \Rightarrow \mu_{13} - \mu_{23} = 0.$

$H_1 : \mu_{13} > \mu_{23}. \Rightarrow \mu_{13} - \mu_{23} > 0.$

$$\hat{\mu}_{13} = \frac{n_{13}}{t_1} = 0.01237$$

$$\hat{\mu}_{23} = \frac{n_{23}}{t_2} = 0.01049$$

$\tilde{\mu}_{13} \sim N[\mu_{13}, CRLB_{13}].$

$$CRLB_{13} = \frac{-1}{E\left[\frac{\partial^2 \ln L}{\partial \mu_{13}^2}\right]} = \frac{-1}{E\left[\frac{-n_{13}}{\mu_{13}^2}\right]} = \frac{\mu_{13}^2}{n_{13}.}$$

$$\therefore \tilde{\mu}_{13} \sim N\left[\mu_{13}, \frac{\mu_{13}^2}{n_{13}}\right]$$

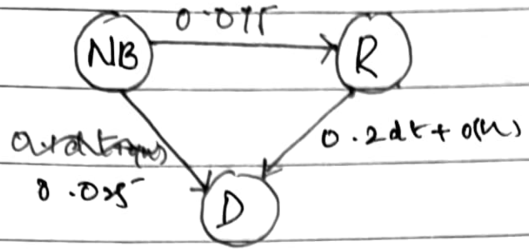
$$\tilde{\mu}_{23} \sim N\left[\mu_{23}, \frac{\mu_{23}^2}{n_{23}}\right].$$

$$\therefore \tilde{\mu}_{13} - \tilde{\mu}_{23} \sim N\left[0, \left(\frac{\mu_{13}^2}{n_{13}} + \frac{\mu_{23}^2}{n_{23}}\right)\right] \text{ under } H_0.$$

$\therefore$ ~~Z~~

Statistic : $\frac{\tilde{\mu}_{13} - \tilde{\mu}_{23}}{\sqrt{CRLB_{13} + CRLB_{23}}} = 1.56.$

10) ii)



iii) FDE :-

$$\frac{d}{dt} P(t) = P(t) \cdot A.$$

$$A = \begin{bmatrix} -0.1 & 0.075 & 0.025 \\ 0 & -0.2 & 0.2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{d}{dt} P_{NB}(t) = -0.1\, P_{NB}(t)$$

$$\frac{d}{dt} P_R(t) = 0.075\, P_{NB}(t) - 0.2\, P_R(t)$$

$$\frac{d}{dt} P_D(t) = P_{NB}(t)\, 0.025 + 0.2\, P_R(t)$$

iii) $$\frac{\frac{d}{dt} P_{NB}(t)}{P_{NB}(t)} = -0.1$$

$$\Rightarrow \int_0^t \frac{d}{ds} \ln P_{NB}(s)\, ds = \int_0^t -0.1\, ds.$$

$$= \ln P_{NB}(t) = -0.1t$$

$$\Rightarrow P_{NB}(t) = e^{-0.1t}$$

$$\frac{d}{dt} P_R(t) = 0.075\, P_R(t)\ \cancel{+} - 0.2\, P_{RR}(t)$$

$$P_{RR}(t) = e^{-0.2t}$$

$$\frac{d}{dt} P_R(t) = 0.075\, P_R(t) - e^{-0.2t}.$$

$$= \frac{d}{dt} P_R(t) - 0.075\, P_R(t) = -e^{-0.2t}$$

$$I.f = e^{\int 0.075\, dt} = e^{0.075t}$$

$$\therefore P_R(t) = \int_0 -e^{-0.2s} \cdot e^{0.075s}\, ds$$

$$= \left[-e^{-0.125s}\right]_0^t = 1 - e^{-0.125t}$$

iv) Required probability is : $e^{-0.1t} + 1 - e^{-0.125t}$