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SRM - I

Q1.

(i) A Stochastic Model has the capacity to handle uncertanities in the inputs applied. Stochastic models posses some ~~into~~ inherent randomness - the same set of parameter values and initial conditions will lead to an ensemble of different outputs.

Advantages of stochastic models:-

1) Stochastic models can reflect real-world economic scenarios that provide a range of possible outcomes you may experience and the relative likelihood of each.

2) By running thousands of calculations, using many different estimates of future economic conditions, stochastic models predict a range of possible future investment results showing the potential upside and downsides of each.

(ii) The time spent in state H before the next visit to S has mean σ^{-1}. Therefore a resonable estimate for σ is the reciprocal of the mean length of each visit :-

$$= \frac{(\text{Number of transitions from H to S})}{(\text{Total time spent in state H up untit the last transition from H to S})}.$$

although it would be equally valid to use the Maximum Likelihood Estimator, which is (Number of transactions from H to S)/(Total time spent in State H). Similarly for ρ.

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Q2.

(i) Consider a Markov chain taking values in the Set

$S = \{ i : i = 0, 1, 2, 3, 4 \}$, where i represent the number of umbredlas in the place where the Actuary currently is (at home or office).

If $i = 1$ and it rains then he take the umbrella move to the other place, where there are already 3 umbrellas and including the one he brings, he will now have 4 umbrellas,

$\therefore P_{1,4} = p$

because p is the probability of rain

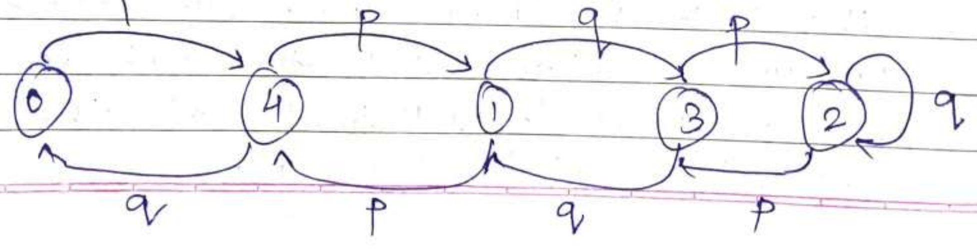
If $i = 1$ but does not rain then he do not take the umbrella, goes to the other place and find 3 umbrellas.

$$P_{1,3} = 1 - p = q.$$

However, if $i = 0$, he must move to other place where 4 umbrellas are kept with probability

$$P_{0,4} = 1$$

Similarly for other states. The process is depicted by the following diagram.



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(ii)
$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q & p \\ 0 & 0 & q & p & 0 \\ 0 & q & p & 0 & 0 \\ q & p & 0 & 0 & 0 \end{pmatrix}$$

(iii) For stationary distribution $\pi P = \pi$

Writing the equations :-

$$\pi(0) = \pi(4).q \quad \text{—} \quad (1)$$
$$\pi(1) = \pi(3)q + \pi(4)p \quad \text{—} \quad (2)$$
$$\pi(2) = \pi(2).q + \pi(3).p \quad \text{—} \quad (3)$$
$$\pi(3) = \pi(1).q + \pi(2)p \quad \text{—} \quad (4)$$
$$\pi(4) = \pi(0) + \pi(1).p \quad \text{—} \quad (5)$$

Substituting 1 in 5

$$\pi(4) = \pi(4).q + \pi(1).p$$

$$\pi(1) = \pi(4).\frac{(1-q)}{p}$$

$$\pi(1) = \pi(4).\frac{p}{p}$$

$$\pi(1) = \pi(4) \quad \text{—} \quad (6)$$

Substituting 6 in 2

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$$\pi(1) = \pi(3)\cdot q + \pi(4)\,p$$
$$\pi(4) = \pi(3)\,q + \pi(4)\,p$$
$$\pi(3) = \pi(4)\cdot\frac{(1-p)}{q}$$
$$\pi(3) = \pi(4) \quad \text{—— } 7$$

Substituting 7 in 3

$$\pi(2) = \pi(2)\cdot q + \pi(3)\cdot p$$
$$\pi(2) = \pi(3)\cdot\frac{p}{(1-q)}$$
$$\pi(2) = \pi(4)\cdot\frac{p}{(1-q)}$$
$$\pi(2) = \pi(4) \quad \text{—— } 8$$

Substituting 6 & 7 in 4

$$\pi(3) = \pi(1)\cdot q + \pi(2)\cdot p$$
$$\pi(4) = \pi(1)\cdot q + \pi(4)\cdot p$$
$$\pi(1) = \pi(4)\cdot\frac{(1-p)}{q}$$
$$\pi(1) = \pi(4)$$

$$\therefore \pi(1) = \pi(2) = \pi(3) = \pi(4) = \frac{\pi(0)}{q}$$

But $\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) = 1$, substituting in terms of $\pi(4)$

$$\pi(4)\cdot q + 4\pi(4) = 1$$

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$$\therefore \pi(4) = \frac{1}{4+q} = \pi(1) = \pi(2) = \pi(3) \text{ and}$$

$$\pi(0) = \frac{q}{4+q}$$

(iv) He will wet every time he happens to be in state 0 and it rains. The chance he is in state 0 is $\pi(0)$. The chance it rains is p.
$\therefore$ Probability that he gets wet.

$P(WET)$ = Probability of being in state 0 and its raining

$$P(WET) = \frac{q}{4+q} \cdot p$$

(v) $$P(WET) = \frac{0.6 \times 0.4}{(4+0.4)} = 5.45\%$$

If he want the chance to be less than 1% then, clearly, he need more umbreallas, So, suppose he need N umbrellas. Set up the Markov chain as above and generalising, it is clear that.

$$\pi(1) = \pi(2) = \pi(3) = \pi(4) \ldots \pi(N) = \frac{\pi(0)}{q}$$

But,
$$\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) \ldots \pi(N) = 1$$
substituting in terms of $\pi(N)$.

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$$\pi(N)\cdot q + N\pi(N) = 1$$

$$\pi(N) = \frac{1}{N+q}$$

$\therefore$ Probability of getting wet $P(wet) = p\,\pi(0)$

$$= \frac{p \times q}{N+q}$$

For $P(Wet) < \frac{1}{100} \Rightarrow \frac{p \times q}{N+q} < \frac{1}{100}$

$N > 100 \times p \times q - q$; given $p = 0.6,\ q = 0.4$

$N > 24 - 0.4 = 23.6$

Thus, it is not worth to keep 24 umbrellas instead of 4 to reduce the probability of getting wet from 6% to 1%. On the days he did not take the umbrella but it starts raining he may buy a cheap (use and throw) umbrella from nearby local shop or take a cab / other mode of transport to office or borrow / share an umbrella from some friend / colleague using the same route or wait for the rain to subside etc.

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Q3.

(i), (ii)

Process	State space	Time Domain
Counting Process	Discrete	Discrete / Continuous
General Random Walk	Discrete or Continuous	Discrete
Poisson Process	Discrete	Continuous
Markov Chain	Discrete	Discrete
Markov Jump	Discrete	Continuous.

(iii)

(a) Number of times the account has been overdrawn since it was opened - Poisson Process / Counting Process.

(b) Status (overdrawn, in credit) of the account on the last day of each month - Markov Jump chain / Counting Process.

(c) Number of direct debits paid since the account was opened - Poisson

(d) Status (overdrawn, in credit) of the account at any time since the account was opened - Markov Jump Process.

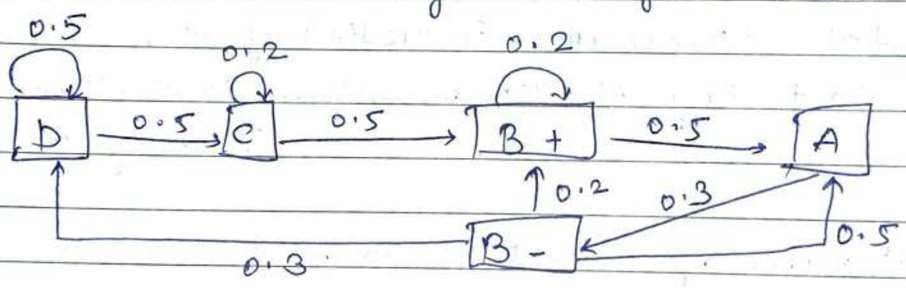
Q4.

(i) Past history is needed to decide where to go in the chain. If a sportsmen is at A and his/her performance he was at the previous year to determine whether he or she drops one or two levels.

(ii) The B level needs to be split into two.

B+ is the level with no reduction in performance parameter last year.
B - is the level with reduction in performance parameter last year.
These levels are for modelling purposes and not the actual levels.

(iii) The Process diagram is given below.



(iv)

	D	C	B+	B-	A
D	0.5	0.5	0	0	0
C	0.3	0.2	0.5	0	0
B+	0	0.3	0.2	0	0.5
B -	0.3	0	0.2	0	0.5
A	0	0	0	0.3	0.7

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(v) Stationary distribution:-

The stationary distribution is the set of probabilities that statisfy the matrix equation

$\pi = \pi P$ with and additional condition $\sum \pi_i = 1$

Written out fully, this set of matrix equation is

$$0.5\pi_D + 0.3\pi_C + 0.3\pi_{B-} = \pi_D \quad \text{— (1)}$$
$$0.5\pi_D + 0.2\pi_C + 0.3\pi_{B+} = \pi_C \quad \text{— (2)}$$
$$0.5\pi_C + 0.2\pi_B + 0.2\pi_{B-} = \pi_{B+} \quad \text{— (3)}$$
$$0.3\pi_A = \pi_{B-} \quad \text{— (4)}$$
$$0.5\pi_B + 0.5\pi_{B-} + 0.7\pi_A = \pi_A \quad \text{— (5)}$$

Substituting (d) in (e)

$$0.5\pi_B + 0.5 \times 0.3\pi_A + 0.7\pi_A = \pi_A$$

$$\pi_{B+} = 0.3\pi_A \quad \text{— (6)}$$

Substituting π_{B+} and π_A in (3) we get

$$0.5\pi_C + 0.2 \times 0.3\pi_A + 0.2 \times 0.3\pi_A = 0.3\pi_A$$
$$\pi_C = 0.36\pi_A$$

Substituting π_{B+}, π_C and π_A in (2)

$$0.5\pi_D + 0.2 \times 0.36\pi_A + 0.3 \times 0.3\pi_A = 0.36\pi_A$$

$$\pi_D = 0.396\pi_A$$

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Substituting all in $\Sigma \pi_i = 1$

$$\pi_A + 0.3\pi_A + 0.3\pi_A + 0.36\pi_A + 0.396\pi_A = 1$$

$$\pi_A = 0.42445$$
$$\pi_{B-} = 0.12733$$
$$\pi_{B+} = 0.12733$$
$$\pi_C = 0.15280$$
$$\pi_D = 0.16808$$

(vi) Long run average contract value is

$$100\%\,\pi_D + 120\%\,\pi_C + 150\%\,(\pi_{B+} + \pi_{B-}) + 175\%\,\pi_A$$
$$= 1.4762 \text{ million USD.}$$

(vii) Let m_i be the number of transitions taken to reach level A from any current level i. Then

$$m_D = 1 + 0.5m_D + 0.5m_C \quad \text{— (a)}$$
$$m_C = 1 + 0.2m_C + 0.5m_B + 0.3m_D \quad \text{— (b)}$$
$$m_{B+} = 1 + 0.2m_B + 0.5m_A + 0.3m_C \quad \text{— (c)}$$
$$m_{B-} = 1 + 0.3m_D + 0.5m_A + 0.2m_{B+} \quad \text{— d}$$
$$m_A = 0$$

from (a) we get,

$$m_D = 2 + m_C$$

substituting in (b) we get,

$$m_C = 1 + 0.2m_C + 0.5m_{B+} + 0.3(2 + m_c)$$
$$m_{B+} = m_C - 3.2$$

Substituting in (c),

$$m_{B+} = 1 + 0.2\, m_{B+} + 0.5\, m_A + 0.3\, m_C$$
$$m_C - 3.2 = 1 + 0.2\,(m_C - 3.2) + 0.3\, m_C$$

$$m_C = 7.12 \text{ years}$$

$$\therefore \quad m_D = 2 + m_2 = 9.12 \text{ years}$$

Q5.

(i) There is an explicit dependence on the past behaviour of $\{Y_j : j < n\}$ in the probability distribution of Y_{n+1}; further, X_n is nothing but the sum of y_j. Hence the Markov property does not hold.

(ii) In part (i), we show that there is explicit dependence on the past behaviour of $\{Y_j : j < n\}$ and hence the Markov property does not hold, this implies that sequence $\{Y_n ; n \geq 1\}$ does not form a Markov chain.

(iii) The transition matrix is given by:

$$\begin{array}{cccccc|}
p & 1-p & 0 & - & - & 0 \\
0 & pe^{-\lambda} & 1-pe^{-\lambda} & 0 & - & - \\
- & 0 & pe^{-2\lambda} & 1-pe^{-2\lambda} & 0 & - \\
- & - & - & - & - & - \\
- & - & - & - & - & - \\
0 & - & - & - & - & -
\end{array}$$

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(iv)

(a) The chain is time homogeneous since the transition probabilities calculated in part (i) is independent of time n.

(b) It is irreducible, since the number of errors can never go down.

(c) There are no recurrent states; hence there can be no stationary distribution.
Alternatively, if a stationary distribution π exists, it has to follow:-

$$\pi_0 p = \pi_0$$
$$\pi_0 (1-p) + \pi_1 p e^{-\lambda} = \pi_1$$
$$\pi_1 (1-pe^{-\lambda}) + \pi_2 pe^{-2\lambda} = \pi_2$$

and so on.
Since $p \leq 1$, we have $\pi_0 = 0$ and then $\pi_1 = 0$,
Hence, no stationary distribution exists.

(v)

Probability of no further error is

$$(pe^{-j\lambda})n = pn \, e^{-nj\lambda}$$

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Q6.

(a) A stochastic model is one that recognizes the random nature of the input components. A model that does not contain any random component is deterministic in nature.

Advantages:

- In a deterministic model, the output is determined once the set of fixed inputs and the relationships between them have been defined. By contrast, in a stochastic model the output is random in nature - like the inputs, which are random variables.

- A deterministic model is really just a special (simplified) case of a stochastic Model.

(b) (i) Assume that the functions $p_{ij}(s,t)$ are continuously differentiable, the transition rates are defined by differentiation with respect to t.

$$\sigma_{ij}(s) = \left[\frac{d}{dt}\left(p_{ij}(s,t)\right)\right]_{t=s}$$

(ii) Since,

$\sum_{j \in S} p_{ij}(s,t) = 1$ where S - state space

We have $\sum_{j \in S} \sigma_{ij}(s) = \sum_{j \in S} \left[d/dt \; p_{ij}(s,t)\right]_{t=s}$

$= d/dt \sum_{j \in S} p_{ij}(s,t)$

$$= \frac{d}{dt}(1)$$

$$= 0$$

Q7.

(a) The process may be expressed as a Markov chain by considering following states. Each state indicates the current number of successive defeats.

State	Number of successive defeats
1	0 - Not defeated last time
2	1 - Defeated in the last match
3	2 - Defeated in the last two matches
4	3 - Defeated in the last three matches
5	4 - Captain sacked

The transition matrix is as follows :-

$$\begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0.7 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) A Markov chain is said to be irreducible if any state j can be reached from any state i. The above process is not irreducible as the captain, once sacked, can never become the captain again.

(c)

(i) The probability of remaining the caption for exactly four matches is given by :-

Match #	#1	#2	#3	#4	Probability
Result	Lose	Lose	Lose	Lose	
Probability	0.3	0.3	0.3	0.3	$= 0.3^4$
					$= 0.0081$

(ii) The probability of remaining the captain for exactly five matches is given by :-

Match #	#1	#2	#3	#4	#5	Probability
Result	Not Lose	Lose	Lose	Lose	Lose	
Probability	0.7	0.3	0.3	0.3	0.3	0.00567

(iii) The probability of remaining the captain for exactly seven matches is given by :-

Match#	#1	#2	#3	#4	#5	#6	#7	Probability
Result	Any	Any	Not Lose	Lose	lose	Lose	Lose	
Probability	1	1	0.7	0.3	0.3	0.3	0.3	0.00567

(iv) The probability of remaining the captain for exactly nine matches is given by :-

Match#	#1	#2	#3	#4	#5	#6	#7	#8	#9	Prob.
Result	Not	all	four	losses	Not lose	Lose	Lose	Lose	Lose	
Probability	= 1-	0.3^4		.	0.7	0.3	0.3	0.3	0.3	
	=	0.9919								0.0056241

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(d) Define N_i = Expected number of matches as a captain given that the current state is i. Since the caption is newly appointed, the variable N in the question is N_1 as defined here.

We have:-

$$N_1 = 1 + 0.7 \times N_1 + 0.3 \times N_2$$
$$N_2 = 1 + 0.7 \times N_1 + 0.3 \times N_3$$
$$N_3 = 1 + 0.7 \times N_1 + 0.3 \times N_4$$
$$N_4 = 1 + 0.7 \times N_1$$

By solving the equation,

$$N_4 = 123.46$$
$$N_3 = 160.49$$
$$N_2 = 171.60$$
$$N_1 = 174.94$$

$\therefore$ $E(N)$ is 174.94 matches

Q8. (i) The score currently stands at 'Tie' whosever wins the next point will move into a 'Lead'. If the player in 'Lead' wins the subsequent point as well, he would win the tie-breaker.

The probability of moving to the next state does not depend on the history prior to entering the state, Markov property holds

The state space is defined as follows

State	Description.
T	Tie
L_F	Federer leads
L_N	Nadal leads
G_F	Federer wins
G_N	Nadal wins

(ii) The transition matrix is set out below:-

$$\begin{bmatrix} 0 & 0.55 & 0.45 & 0 & 0 \\ 0.45 & 0 & 0 & 0.55 & 0 \\ 0.55 & 0 & 0 & 0 & 0.45 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) The chain is reducible as it has two absorbing states - G_F and G_N.

Absorbing states have no period and the other three states have a period of 2. Thus, the chain is not a-periodic.

(iv) After two points from the tie, the tie-breaker would either be completed or be back to tie again.

Probability of Federer winning the first point × Probability of Nadal winning the second point + Probability of Nadal winning the first point

$= 0.55 \times 0.45 + 0.45 \times 0.55 = 0.495.$

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We need to find number of such cycles of returning to tie such that

$$0.495^N = 1 - 0.95$$

$$N = \frac{\ln 0.05}{\ln 0.495} = 4.26.$$

$\because$ The game can finish in cycles of two points, the required number of cycles is 5 i.e 10 points.

(v) After two points :-

a. Nadal may have won the tie-breaker (probability of 0.2025 i.e. 0.45^2); or

b. Federer may have won the tie-breaker (probability of 0.3025 i.e. 0.55^2) or

c. Tie-breaker may have come back to tie (probability of ~~0.4~~ 0.495).

Let F_T be the probability that Federer wins the tie-breaker that is presently tied

Let N_T be the probability that Nadal wins the tie-breaker that is presently tied.

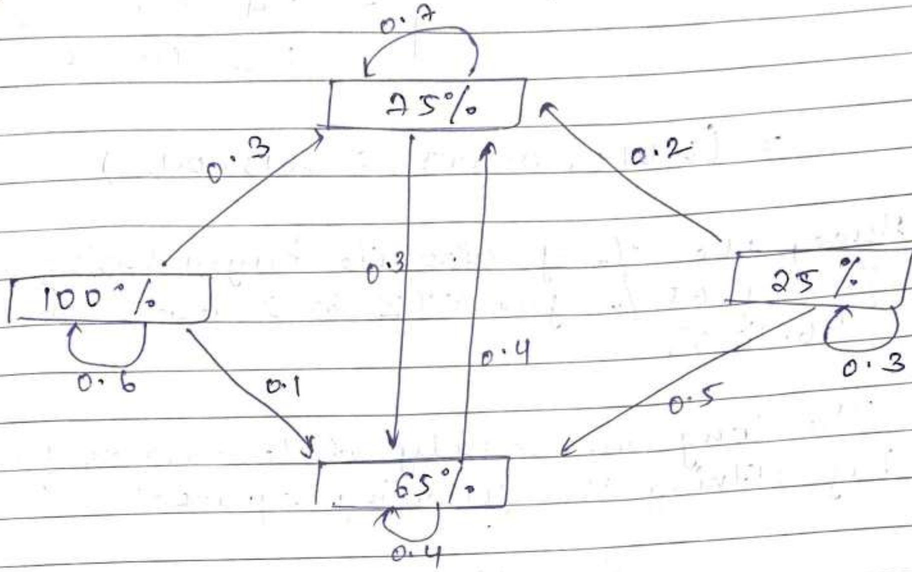
$$N_T = 0.2025 + 0.495 \times N_T$$

$$N_T = \cancel{0.2} \; 0.401$$

$$F_T = 0.3025 + 0.495 \times F_T$$
$$F_T = 0.599$$

Q9.

(a) The state transition diagram is set out below:-



(b) We are to calculate for the % of corporate buyer having a target % for XYZ of 65%
- In two years time
- Over the long-run

The state of the system after one year $S1 = S0P$

$$(0.05, 0.30, 0.45, 0.20)\begin{bmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{bmatrix} = (0.03, 0.445, 0.375, 0.15)$$

Hence, the state of the system in 2 year time
$$S2 = S1\,P.$$

$$(0.03, 0.445, 0.375, 0.15)\begin{bmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{bmatrix}$$

$$= (0.018, 0.5005, 0.3613, 0.12)$$

Hence, the % of corporate buyer having a target % of 65% for XYZ in 2 years time is 36.15%.

The long run steady state can be found by solving the following equation:-

$$S = SP$$

i.e.
$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)P$$

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)\begin{bmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{bmatrix}$$

$$x_1 = 0.6x_1$$
$$x_2 = 0.3x_1 + 0.7x_2 + 0.4x_3 + 0.2x_4$$
$$x_3 = 0.1x_1 + 0.3x_2 + 0.4x_3 + 0.5x_4$$
$$x_4 = 0.2x_3 + 0.3x_4$$
$$\therefore\ x_1 + x_2 + x_3 + x_4 = 1$$

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Now from the first equation above,

$$0.4x_1 = 0$$

So

$$x_1 = 0$$

Substituting this into the other equations above and rearranging we get

$$0.3x_2 = 0.4x_3 + 0.2x_4 \quad \text{--- (1)}$$
$$0.6x_3 = 0.3x_2 + 0.5x_4 \quad \text{--- (2)}$$
$$0.7x_4 = 0.2x_3 \quad \text{--- (3)}$$

$$x_2 + x_3 + x_4 = 1$$

One of the above equations is redundant

$$\therefore x_1 = 0 \quad x_2 = 0.5423 \quad x_3 = 0.3559$$
$$x_4 = 0.1017.$$

(c) The steady state proportion of customers having a 65% target allocation for XYZ Ltd is 35.59%. It is noted that the initially this proportion is 45% and it quickly drops to:-

- 37.5% in one year's time; and
- 36.2% in two year's time.

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Q1.

(i) If each room is represented by the state, then the transition matrix P for this Markov chain is as follows:-

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

(ii) The chain is irreducible, because it is possible to go from any state to any other state. However, it is not aperiodic, because for any even n, $P_{6,1}{}^{n}$ will be zero and for any odd n $P_{6,5}{}^{n}$ will also be zero. This means that there is no power of P that would have all its entries strictly positive.

(iii) For P to be stationary, $\pi P = P$
Perform matrix multiplication and show that πP is equal to P

~~(v) Let,~~

(iv) We find from π that the mean recurrence time (i.e. the expected time to return) for the room 1 is $\frac{1}{\pi(1)} = 12$.