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SRM 4 Assignment 2
Unit 3 & 4
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1] $Y_t^2 - \beta_1 e_t^2 Y_{t-1}^2 = 2(Y_t - \beta_1 e_t^2 Y_{t-1})\mu - (1 - \beta_1 e_t^2)\mu^2 + \beta_0 e_t^2$

(i) $\text{cov}[Y_t, Y_{t-s})$

$Y_t^2 - 2Y_t\mu + \mu^2 = e_t^2(\beta_0 + \beta_1 Y_{t-1}^2 - 2\beta_1 Y_{t-1}\mu + \beta_1\mu^2)$

$(Y_t - \mu)^2 = e_t^2(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)$

$Y_t = \mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{1/2}$

$E[Y_t] = E[\mu] + E[e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}$

e_t & Y_{t-1} are independent

$E[Y_t] = E[\mu] + E[e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}]$

$E[Y_t] = \mu + 0 * E[\ldots]$

$E[Y_t] = \mu$

$\text{cov}[Y_t, Y_{t-1}) = E(Y_t Y_{t-s}) - E(Y_t)E(Y_{t-s})$

$= E[(\mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})(\mu + e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}] - \mu^2$

$= E[\mu^2 + \mu e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5} + \mu e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5} + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}$

$e_{t-s}(\beta_0 + \beta_1(y_{t-s-1} - \mu)^2)^{0.5}) - \mu^2$

~~$\mathrm{Cov}(y_t, y_{t-1}) = \mu^2 + \mu E(e_t) E((\beta_0 + \beta_1(y_{t-s-1} - \mu)^2)$~~
~~$+ e$~~

$\mathrm{Cov}(y_t, y_{t-1}) = \mu^2 + \mu E(e_t) E((\beta_0 + \beta_1(y_{t-1} - \mu)^2)^{0.5})$
$+ \mu E(e_{t-s}) E((\beta_0 + \beta_1(y_{t-s-1} - \mu)^2)^{0.5}) +$
$E(e_t) E(e_{t-s}) E(((\beta_0 + \beta_1(y_{t-1} - \mu)^2)^{0.5})$
$((\beta_0 + \beta_1(y_{t-s-1} - \mu)^2)^{0.5}) - \mu^2$

$= \mu^2 - \mu^2$

$\mathrm{Cov}(y_t, y_{t-1}) = 0$

ii) $\mathrm{Var}(y_t / y_{t-1}) = \mathrm{Var}(e_t)\,\mathrm{Var}(\beta_0 + \beta_1(y_{t-1} - \mu)^2)$
$= \beta_0 + \beta_1(y_{t-1} - \mu)^2$

Var of y_t is independent of y_{t-1}
$\therefore$ Var of y_t will depend on y_{t-s}
y_t & y_{t-s} are dependent.

(iii) $\Delta X_t = X_t - X_{t-1}$

$E(\Delta X_t) = E(X_t) - E(X_{t-1})$
$= E(0.5 y_t + 0.3t + 0.1) - E(0.5 y_{t-1} +$
$0.3(t-1) + 0.1)$
$= 0.5\mu + 0.3t + 0.1 - 0.5\mu - 0.3(t-1) - 0.1$
$= 0.3$

Mean is independent of t & constant

$$\begin{aligned}\mathrm{Cov}(\Delta X_t, \Delta X_{t-s}) &= \mathrm{Cov}(X_t - X_{t-1}, X_{t-s} - X_{t-s-1}) \\ &= \mathrm{Cov}(0.3 + Y_t - Y_{t-1}, 0.3 + Y_{t-s} - Y_{t-s-1}) \\ &= \mathrm{Cov}(Y_t - Y_{t-1}, Y_{t-s}, - Y_{t-s-1}) \\ &= \mathrm{Cov}(Y_t - Y_{t-s}) - \mathrm{Cov}(Y_t - Y_{t-s-1}) \\ &\quad - \mathrm{Cov}(Y_{t-1} - Y_{t-s}) + \mathrm{Cov}(Y_{t-1}, Y_{t-s-1}) \\ &= 0\end{aligned}$$

Autocorrelation function is constant & so difference of X_t is stationary

Q2] i) Characteristic eq: $1 - z - 0.5z^2 + 0.5z^3 = 0$

$X_t - 0.5X_{t-2} = Z_t + 0.3Z_{t-1}$

as $X = (1-B)Y$

. ARIMA (2, 1)

$\therefore$ Y is ARIMA (2, 1, 1)

ii) Characteristic of polynomial of X is $(1 - 0.5z^2)$

Roots = $\pm\sqrt{2} > 1$

$\therefore$ Process is stationary

(iii) Model eq: $X_t = 0.5X_{t-2} + Z_t + 0.3Z_{t-1}$

$$\begin{aligned}\mathrm{Cov}(X_t, Z_t) &= \mathrm{Cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_t) \\ &= 0 + \sigma^2 + 0 \\ &= \sigma^2\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}(X_t, Z_{t-1}) &= \mathrm{Cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_{t-1}) \\ &= 0.3\sigma^2\end{aligned}$$

$\mathrm{cov}(X_t, Z_{t-2}) = \mathrm{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_{t-2})$
$= 0.5\sigma^2$

$\gamma(0) = \mathrm{cov}(X_t, X_t)$
$= \mathrm{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_t)$
$= 0.5\gamma(2) + \sigma^2 + 0.09\sigma^2$
$= 0.5\gamma(2) + 1.09\sigma^2$

$\gamma(1) = \mathrm{cov}(X_t, X_{t-1})$
$= \mathrm{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-1})$
$= 0.5\gamma(1) + 0 + 0.3\sigma^2$

$\gamma(2) = \mathrm{cov}(X_t, X_{t-2})$
$= \mathrm{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-2})$
$= 0.5\gamma(0)$

$\gamma(k) = \mathrm{cov}(X_t, X_{t-k})$
$= \mathrm{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-t})$
$= 0.5\gamma(k-2)$ for $k > 2$

$\gamma(0) = 0.25\gamma(0) + 1.09\sigma^2$
$\gamma(0) = \frac{1.09\sigma^2}{75}$

$\gamma(1) = \frac{3\sigma^2}{0.5}$ $\rho_1 = \frac{\gamma(1)}{\gamma(0)} = \frac{45}{109}$

$\rho(k) = 0.5\,\rho(k-2)$

$\rho(k) = 0.5^{|k|/2}$

$= \frac{45}{109} (0.5)^{\frac{|k|-1}{2}}$ if $|k|$ is odd

3] i) a) The process is $(1-0.4B-0.2B^2)Y_t = (1+0.025B) Z_t + 0.016$

characteristic eq : $1-0.4z-0.2z^2=0$

no root $=1$

$\therefore d=0$

ARIMA(2,0,1) process.

b) $(1-0.4-0.2)E(Y_t) = 0.016$

$E(Y_t) = \frac{0.016}{0.4} = 4\%$.

c) 2 roots are $-1 \pm \sqrt{6}$

1.4495 and $-3.4495 > 1$

Y_t is a stationary process.

(ii) a) doubt

b) Inspection of graph of time plot of residuals

- pattern such as clusters or uneven fluctuations which indicate inadequate fit.

Inspection of sample autocorrelation fn of residuals.

- If too many residual values lie beyond range $\pm 2/\sqrt{N}$ indicates poor fit or too few parameters.

4) i) AR(2) process

Model can be written as $(1+\alpha B-\alpha^2B^2)Y_t = Z_t$

characteristic eq: $1+\alpha x-\alpha^2x^2=0$

$$x = \frac{1\pm\sqrt{5}}{2\alpha}$$

$\therefore |\alpha|>1$ should be.

$$\frac{\sqrt{5}-1}{2|\alpha|}>1 \quad \& \quad \frac{\sqrt{5}+1}{2|\alpha|}>1$$

$$|\alpha| < \frac{\sqrt{5}-1}{2}$$

ii) $Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$

Yule walker eqs are:

$$\operatorname{cov}(Y_t, Y_t) = \gamma_0$$
$$= -\alpha\gamma_1 + \alpha^2\gamma_2 + \sigma^2$$

$$\operatorname{Cov}(Y_t, Y_{t-1}) = -\alpha\gamma_0 + \alpha^2\gamma_1 = \gamma_1$$

$$\operatorname{Cov}(Y_t, Y_{t-2}) = \gamma_2$$
$$= -\alpha\gamma_1 + \alpha^2\gamma_0$$

$$\gamma_1 = -\frac{\alpha\gamma_0}{1-\alpha^2}$$

$$\gamma_2 = -\alpha\left[-\frac{\alpha\gamma_0}{1-\alpha^2}\right] + \alpha^2\gamma_0$$

$$= (2\alpha^2-\alpha^4)\frac{\gamma_0}{1-\alpha^2}$$

$$\gamma_0 = \frac{\sigma^2(1-\sigma^2)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$

$$\gamma_1 = \frac{\sigma^2(-\alpha^1)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$

$$\gamma_2 = \frac{\sigma^2(2\alpha^2-\alpha^4)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$

5) i) The model is $(X_n - X_{n-1}) = \alpha(X_{n-1} - X_{n-2}) + e_n$

$$\alpha = \rho_1 = \frac{\sum_{i=3}^{200}(X_i - X_{i-1})(X_{i-1} - X_{i-2})}{\sum_{i=2}^{200}(X_i - X_{i-1})^2}$$

$$\alpha = \frac{587.83}{936.49} = 0.6277$$

$$\gamma_0 = \frac{1}{200}\sum_{i=2}^{200}(X_i - X_{i-1})^2 = \frac{936.49}{200} = 4.6825$$

$$\gamma_0 = \frac{\sigma^2}{1-\rho_1^2} = \frac{\sigma^2}{1-\alpha^2}$$

$$\hat{\alpha}^2 = (1-\hat{\alpha}^2)\hat{\gamma}_0 = (1-0.6277^2)\cdot 4.6825 = 2.8376$$

ii) Forecast of X_{201}

$$(\hat{X}_{200}(1) - X_{200}) = \hat{\alpha}(X_{200} - X_{199})$$

$$0.6277(1.93 - 0.82) = 0.6967$$

$$\hat{X}_{200}(1) = X_{200} + 0.6967$$

$$= 1.93 + 0.6967$$

$$= 2.6267$$

6) i) Purpose of practical time series analysis is :
- description of data
- construction of model which fits the data
- forecasting future values of process
- for vector time series, investigating connections btwn 2 or more observed processes with the aim of using values of some of processes to predict those of others.
- deciding whether the process is out of control

ii) AR(1) & Random walk are Markov
MA(1) & AR(2) are not Markov.

iii) a) $E(X_t) = a + bt + E(Y_t)$
Since Y_t is stationary ; $E(Y_t) = c$ that does not depend on t
$E(X_t) = a + bt + c$ which depends on t
~~If f~~ $\therefore X_t$ cannot be stationary

b) $E[\nabla X_t] = E[X_t] - E[X_{t-1}] = a + bt + c - (a + b(t-1) + c) = b$
Thus mean does not depend on t.
$Cov[\nabla X_t, \nabla X_s] = Cov[X_t - X_{t-1}, X_s - X_{s-1})$
$= \cancel{Cov(b + Y_t - Y_{t-1}, b + Y_s - Y_{s-1})}$
$= Cov(b + Y_t - Y_{t-1}, b + Y_s - Y_{s-1})$

$= Cov(Y_t - Y_{t-1}, Y_s - Y_{s-1})$

$= Cov(Y_t, Y_s) - Cov(Y_t, Y_{s-1}) - Cov(Y_{t-1}, Y_s) + Cov(Y_{t-1}, Y_{s-1})$

$= C_{Y(t-s)} - C_Y(t-s+1) - C_Y(t-s-1) + C_Y(t-s)$

∇X_t is stationary

iv) $X_1(t) - X_2(t) = \alpha[X_1(t-1) - X_2(t-1)] + b[X_2(t-1) - X_1(t-1)] + [e_1(t) - e_2(t)]$

$= (a-b)[X_1(t-1) - X_2(t-1)] + [e_1(t) - e_2(t)]$

Let $Y(t) = X_1(t) - X_2(t)$

$e(t) = e_1(t) - e_2(t)$

$\because$ $Y(t)$ must be an AR(1) process.

$Y(t)$ is stationary as long as $|a-b| < 1$

Q7] 1) a) Auto Regressive Process (AR)

→ An auto regressive process with order p is a sequence of rv (X_t) defined consecutively by the rule.

$X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \dots + \alpha_p(X_{t-p} - \mu) + e_t$

Thus, AR(p) model attempts to explain the current value of X as a linear combination of past values with some additional externality generated rv

b) Moving Average Process (MA)
A MA process with order q is a sequence defined by
$x_t = \mu + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}$
MA(q) model explains the relation between X_t as an indirect effect, arising from the fact that the current value of process results from recently past random errors as well as the current one.

(c) Autoregressive Moving Average process (ARMA)
Two basic processes (AR & MA) can be combined to give an auto regressive moving average (ARMA) process. The eq
$$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \cdots + \alpha_p (X_{t-p} - \mu) + e_t + \cancel{e_t} + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}$$

(iii) a) MA(1) process & hence stationary
∴ ARIMA(0,0,1)

b) ARMA (2,3) process
Process cannot be differenced, so to clarify we need to check that it is stationary
$$(1 - 1.4B^2) X_t = e_t + 0.5 e_t$$
$$\phi(\lambda) = 1 - 1.4\lambda^2 = 0$$
$$\lambda = \pm 0.8452$$
both roots < 1
non stationary

c) ARMA(2,1) process

$$X_t - 1.4X_{t-1} + 0.4X_{t-2} = e_t + e_{t-1}$$

$$(X_t - X_{t-1}) - 0.4(X_{t-1} - X_{t-2}) = e_t + e_{t-1}$$

$$\nabla X_t - 0.4\nabla X_{t-1} = e_t + e_{t-1}$$

ARIMA(1,1,1)

(iii) Process is stationary as it is sum of stationary white noise terms.

Autocorrelation fn is

$$\gamma_0 = Cov(X_t, X_t) = Var(X_t)$$

$$= Cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3},$$
$$e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3})$$

$$= \sigma^2 + 0.25^2\sigma^2 + 0.5^2\sigma^2 + 0.25^2\sigma^2$$

$$= 1.375\sigma^2$$

$$\gamma_1 = Cov(X_t, X_{t-1})$$

$$= Cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-1} +$$
$$0.25e_{t-2} + 0.5e_{t-3} + 0.25e_{t-4})$$

$$= 0.25\sigma^2 + 0.5\times0.25\times\sigma^2 + 0.5\times0.25\times\sigma^2$$

$$= 0.5\sigma^2$$

$$\gamma_{\pm 2} = Cov(X_t, X_{t-2})$$

$$= Cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-2} +$$
$$0.25e_{t-3} + 0.5e_{t-4} + 0.25e_{t-5})$$

$$= 0.5\sigma^2 + 0.25^2\sigma^2$$

$$= 0.5625\sigma^2$$

$\gamma_{\pm 3} = Cov(X_t, X_{t-3})$

$= Cov(e_t + 0.25 e_{t-1} + 0.5 e_{t-2} + 0.25 e_{t-3}, e_{t-3} + 0.25 e_{t-4} + 0.5 e_{t-5} + 0.25 e_{t-6})$

$= 0.25\sigma^2$

$\gamma_{\pm k} = 0$ for $|k| > 3$

$\rho_0 = 1$

$\rho_{\pm 1} = 0.364$

$\rho_{\pm 2} = 0.409$

$\rho_{\pm 3} = 0.183$

$\rho_{\pm k} = 0$ for $|k| > 3$

8] (i) $X_t = A_t + B_t$

$A_t = 0.5 A_{t-1} + 0.5 B_{t-1} + e_1^A$

$B_t = 0.7 A_{t-1} - 0.7 A_{t-2} + e_1^B$

using matrix notation we get,

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-1} \\ B_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-2} \\ B_{t-2} \end{pmatrix} + \begin{pmatrix} e_1^{(A)} \\ e_1^{(B)} \end{pmatrix}$$

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{bmatrix} Y_{t-1} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} Y_{t-2} + \begin{pmatrix} e_1^{(A)} \\ e_1^{(B)} \end{pmatrix}$$

Eigen values $\lambda^2 - 0.5\lambda - 0.35 = 0$

$|\lambda| < 1$

For second matrix $|\lambda| < 1$

$\therefore$ Y_t is stationary

(ii) a) $X_t = (\alpha+1) X_{t-1} - (\alpha + 0.25\alpha^2) X_{t-2} + 0.25\alpha^2 X_{t-3} + e_t$.

$X_t - X_{t-1} = \alpha(X_{t-1} - X_{t-2}) - 0.25\alpha^2 (X_{t-2} - X_{t-3}) + e_t$

$Y_t = \alpha Y_{t-1} - 0.25\alpha^2 Y_{t-2} + e_t$

$\cancel{Y_t = \alpha Y_{t-1} - 0.}$ X_t is ARIMA $(2,1,0)$

$(1 - \alpha B + 0.25\alpha^2 B^2) Y_t = e_t$.

characteristic eq : $1 - \alpha\lambda + 0.25\alpha^2\lambda^2 = 0$

$\lambda = \frac{2}{\alpha}$

To be stationary, $|\lambda| > 1$

$|\alpha|$ should be less than 2

X_t is ARIMA $(2,1,0)$ process with $|\alpha| < 2$

b) $Cov(Y_t, e_t) = Cov(e_t, e_t) = \sigma^2$

$\gamma_0 = Cov(Y_t, Y_t) = \alpha\gamma_1 + 0.25\alpha^2\gamma_2 + \sigma^2$

$\gamma_1 = \alpha\gamma_0 - 0.25\alpha^2\gamma_1$

$\gamma_2 = \alpha\gamma_1 - 0.25\alpha^2\gamma_0$

$\gamma_k = \alpha\gamma_{k-1} - 0.25\alpha^2\gamma_{k-2}$

$\gamma_1 = \frac{\alpha\gamma_0}{1 + 0.25\alpha^2}$

$$\gamma_0 = \alpha\gamma_1 - 0.25\,\alpha^2(\alpha\gamma_1 - 0.25\,\alpha^2\gamma_0) + \sigma^2$$

$$(1-(0.25\,\alpha^2)^2)\gamma_0 = \frac{\alpha^2(1-0.25\,\alpha^4)}{1+0.25\,\alpha^2}\gamma_0 + \sigma^2$$

$$\gamma_0 = \frac{1+0.25\,\alpha^2}{(1-0.25\,\alpha^2)^3}\sigma^2$$

$$\gamma_1 = \frac{\alpha}{1+0.25\,\alpha^2}\left(\frac{1+0.25\,\alpha^2}{(1-0.25\,\alpha^2)^3}\right)\sigma^2$$

$$= \frac{\alpha}{(1-0.25\,\alpha^2)^3}\sigma^2$$

for $k \geq 2$;

$$\gamma_k = \alpha\gamma_{k-1} - 0.25\,\alpha^2\gamma_{k-2}$$

Substituting,

$$\gamma_k = \alpha(0.5\,\alpha)^{k-1}[\lambda_1 + (k-1)\lambda_2] - 0.25\,\alpha^2 (0.5\,\alpha)^{k-2}[\lambda_1 + (k-2)\lambda_2]$$

$$\gamma_k = \lambda_1\alpha^k(0.5)^{k-1}\left[1 - \frac{0.25}{0.5}\right] + \lambda_2\alpha^k(0.5)^{k-1}\left[(k-1) - \frac{1}{2}(k-2)\right]$$

$$\gamma_k = \lambda_1\alpha^k(0.5)^k = k\lambda_2\alpha^k(0.5)^k$$

$$\lambda_1 + k\lambda_2 = (0.5\,\alpha)^{-k}\gamma_k$$

$$\lambda_2 = \frac{1}{0.5\,\alpha}\gamma_1 ; \quad \lambda_1 = \frac{\sigma}{(1-0.25\,\alpha^2)^2}$$

c) $\alpha = 0.04$

$y_t = 0.04\, y_{t-1} - 0.0004\, y_{t-2} + e_t$

$y_t = x_t - x_{t-1}$ / $x_t = y_t + x_{t-1}$

$x_{51} = y_{51} + x_{50}$

$x_{52} = y_{52} + x_{51}$

Forecasted values are,

$x_{51} = y_{51} + x_{50}$

$x_{52} = y_{52} + x_{51}$

where $y_{51} = 0.04(x_{50} - x_{49}) - 0.0004(x_{49} - x_{48})$

$y_{52} = 0.04\, y_{51} - 0.0004(x_{50} - x_{49})$

10) i) $p = 1,\ q = 4$

Follow ARMA(1,4)

ii) Non linear non stationary time series model includes

Bilinear models are those that exhibit bursty behaviour:

$x_n - \alpha(x_{n-1} - \mu) = \mu + e_n + \beta e_{n-1} + b(x_{n-1} - \mu)e_{n-1}$

Threshold AR model are used to model cyclical behaviour

$x_n = \mu + \alpha_1(x_{n-1} - \mu) + e_n$ -- if $x_{n-1} \leq d$

$x_n = \mu + \alpha_2(x_{n-1} - \mu) + e_n$ if $x_{n-1} \geq d$

$x_t = \mu + \alpha(x_{t-1} - \mu) + e_t$.

$$x_t = \mu + e_t \sqrt{\alpha_0 + \sum \alpha_k (x_{t-k} - \mu)^2}$$

(iii) $X_t \sim MA$;

$X_t = e_t + \beta e_{t-1}$ where $e_t \sim (0, \sigma^2)$

$Var(X_t) = Var(e_t + \beta e_{t-1})$

$Var(e_t) + \beta^2 Var(e_{t-1})$

$(1+\beta^2)\sigma^2$

$\Delta Y_t = (0.6 + 0.3t + X_t) - (0.6 + 0.3(t-1) + X_{t-1})$

$= 0.3 + (X_t - X_{t-1})$

Hence $Var(\Delta Y_t) = Cov(X_t - X_{t-1}, X_t - X_{t-1})$

$2\gamma_x(0) - \gamma_x(1) - \gamma_x(1)$

$\gamma_x(0) = (1+\beta^2)\sigma^2$

$\gamma_x(1) = \gamma_x(-1) = cov(e_t + \beta e_{t-1}, e_t + \beta e_{t-1})$

$= \beta\sigma^2$

$Var(\Delta Y_t) = 2(1+\beta^2)\sigma^2 - 2\beta\sigma^2$

$= 2\sigma^2(1 - \beta + \beta^2)$

$Var(\Delta Y_t) - Var(X_t)$

$(2 - 2\beta + 2\beta^2)\sigma^2 - (1+\beta^2)\sigma^2$

$(1 - 2\beta + \beta^2)\sigma^2$

$(1-\beta^2)\sigma^2 > 0$

Hence, sd of first difference of Y_t is higher than that of X_t.

11) i) Purposes of practical time series analysis:

→ description of data

→ construction of model that fits the data

→ forecasting future values of the process

→ deciding whether process is out of control, requiring action.

→ for vector time series, investigating connections between 2 or more observed process with aim of using values of some of processes to predict those of others.

ii) Univariate TS: sequence of observations recorded at regular intervals. State space is continuous, but time set is discrete. Such series may follow a pattern to some extent, for eg. possessing a trend or seasonal component.

Invertibility, A time series process X is invertible if we can write the white noise e_t as a convergent sum of X terms. It is a desirable characteristic since it enables us to calculate residual terms & hence analyse the goodness of fit of particular model.

(iii) Following methods can be used to remove seasonal variation:

→ seasonal differencing
→ Method of moving average.

Following methods to remove any linear trend;

→ least square trend removal
→ differencing

iv) a) $Y_t - 0.6Y_{t-1} = 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}$

$(1-0.6B)Y_t = 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}$

$Y_t = (1-0.6B)^{-1}(0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2})$

$Y_t = \sum_0^\infty 0.6^i B^i (0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2})$

$= \frac{0.5}{1-0.6} + 0.2\sum_-^\infty 0.6^i e_{t-i} + 0.7\sum 0.6^i e_{t-1-i} +$

$0.3\sum 0.6^i e_{t-2-i}$

$= 1.25 + 0.2(1e_t + 0.6e_{t-1}) + 0.7e_{t-1} + (0.3 * 0.7 \times 0.6 + 0.2 \times 0.6^2)\sum 0.6^{i-2} e_{t-i}$

$= 1.25 + 0.2e_t + 0.19e_{t-1} + 1.192\sum 0.6^{i-2} e_{t-i}$

12) i) lack of stationarity :

- lack of stationarity may be caused by presence of deterministic effects in quantity being observed.
- If process observed is integrated version of more fundamental process.
- ~~for eg. A company~~.

ii) a) $(1-B)^d \phi(B)(X_t - \mu) = \theta(B) e_t$

where $\phi(B) = 1 - \sum_i B^i \alpha_i$

$\theta(B) = \sum_i^q B^j \beta_j$

$$(\nabla^d X_t - \mu) = \sum \alpha_i (\nabla^d X_{t-1} - \mu) + e_t + \sum_{1}^{q} \beta_j e_{t-j}$$

(b) Main steps in Box-Jenkins methodology

- Tentative identification of a model from ARIMA class
- Estimation of parameters in identified model
- Diagnostic checks.

(c) If sample autocorrelation coefficients decay slowly from 1 then this indicates that further differencing required.

Not the case when $d=1$, which means that differencing of original series required once hence $d=1$.

further correct value of d minimises sample variance.

d) 1) $X_t = 0.8 e_{t-1} + e_t$

MA (1) → stationary

ARIMA (0, 0, 1)

2) $X_t = 2X_{t-2} + e_t + 0.5 e_{t-3}$

ARMA (2,3). cannot be differenced further

∴ classify as ARIMA (2, 0, 3)

3) $X_t = 1.5 X_{t-1} + 0.2 X_{t-2} + e_t + e_{t-1}$
ARMA (2,1)

13) i) Stochastic process is weakly stationary if it has constant mean & covariance is constant for each fixed lag.

(ii) moving Avg process is $X_n = Z_n + \beta Z_{n-1}$
$E(X_n) = (1+\beta)\mu$
Variance
$Var(X_n) = Var(Z_n + \beta Z_{n-1})$
$Var(X_n) = Var(1+\beta^2)\sigma^2$

Covariance
$Cov(X_n, X_{n+1}) = Cov(Z_n + \beta Z_{n-1}; Z_{n+1} + \beta Z_n)$
$= \beta\sigma^2$

for $m > 1$;
$Cov(X_n, X_{n+m}) = Cov(Z_n + \beta Z_{n-1}, Z_{n+m} + \beta Z_{m+n-1})$
$= 0$
Covariance at higher lags is 0 since there is no overlap between the Z's.

(iii) $(1 - 1.5B + 5B^2) X = Z$
Not Stationary

iv) ARIMA (1,1,0).