ASSIGNMENT 2

Q1. $X \rightarrow$ claim size of home insurance

$Y \rightarrow$ claim size of car insurance

$X \sim N(800, 100^2)$, $Y \sim N(1200, 300^2)$

$$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$$
$$= P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$$

$$Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) \sim N(((3 \times 1200) - 41800), (300^2 \times 3 + 400^2 \times 4))$$
$$\sim N(400, 310000)$$

$$P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right) = P(Z > 0.71842)$$
$$= 1 - P(Z < 0.71842)$$
$$= 1 - 0.76424$$
$$= 0.23576$$

Q2. Let $X \rightarrow$ claim amounts

$X \sim \text{Gamma}(6, 2)$

$$P(N = n) = \binom{n+2}{n} (0.9)^3 (0.1)^n$$

(i) $Y = X_1 + X_2 + \dots + X_n$

$$M_Y(t) = E(e^{tY})$$
$$= E(e^{t(X_1 + X_2 + \dots + X_n)})$$
$$= E(e^{tX_1} \cdot e^{tX_2} \dots e^{tX_n})$$
$$= E(e^{tX_1}) \cdot E(e^{tX_2}) \dots E(e^{tX_n})$$
$$= [E(e^{tX})]^n \quad (\because X_1, X_2, \dots \text{ are iid})$$
$$= [M_X(t)]^n$$
$$= E(e^{(\ln M_X(t))^n})$$
$$= E(e^{N \ln M_X(t)})$$
$$= M_N(\ln M_X(t))$$
$$= \left(\frac{p e^{\ln M_X(t)}}{1 - q e^{\ln M_X(t)}}\right)^n$$

$P(N=n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$

$\Rightarrow p = 0.9,$

$r = 3$

$\cancel{\Rightarrow n+r} \Rightarrow n = 3 - 1 = 2$

$\therefore M_Y(t) = \left(\dfrac{0.9 e^{\ln M_X(t)}}{1 - 0.1 e^{\ln M_X(t)}}\right)^2$

Also, $M_X(t) = \left(1 - \dfrac{t}{2}\right)^{-6}$

$= \left(\dfrac{2-t}{2}\right)^{-6}$

$\therefore$ MGF of Y, $M_Y(t) = \left(\dfrac{0.9\left(\frac{2-t}{2}\right)^{-6}}{1 - 0.1\left(\frac{2-t}{2}\right)^{-6}}\right)^2$

$= \left(\dfrac{0.9(2-t)^{-6}}{2^6 - 0.1(2-t)^{-6}}\right)^2$

ii) $V[Y] = V[X_1] + V[X_2] + \dots + V[X_n]$

$= n V[X]$

$= 2 \times \dfrac{6}{2^2}$

$= 3$

$\therefore$ standard deviation of $Y = \sqrt{3}$

Q3. i) $M_X(t) = E(e^{tX})$

$f_X(x) = \lambda e^{-\lambda(x-\alpha)}$ — ①

For exponential distribution,

$f_Y(y) = \lambda e^{-\lambda y}$ — ②

Comparing ① & ②,

$y = x - \alpha$

$\therefore X \sim \exp(\lambda)$ and $\alpha = 1$

ii) Mean $= \frac{1}{\lambda}$

Variance $= \frac{1}{\lambda^2}$

Q4. $G_V(t) = \left(\frac{p}{1-qt}\right)^k$

$p + q = 1$

$G_V(t) = E(t^V)$

$= \sum t^v \cdot P(V=v)$

$= t^0 \cdot P(V=0) + t^1 \cdot P(V=1) + t^2 \cdot P(V=2) + \dots$

$\cancel{P(V=4) = \frac{\sqrt{k+4}}{\sqrt{5}\sqrt{k}} p^k q^k}$

$G_V(t) = p^k (1-qt)^{-k}$

$\therefore$ $P(V=4)$ would be the 5th term of the series expansion divided by t^4.

$\therefore P(V=4) = \frac{p^k \times {}^{-k}C_4 \, 1^{(-k-4)} (qt)^4}{t^4}$

$$= p^k \times {}^{-k}C_4 \times q^4$$

$$= p^k q^4 \times \frac{-k!}{(-k-4)!\,4!}$$

$$= \frac{p^k q^4}{24} \frac{-k!}{-(k+4)!}$$

$$= \frac{p^k q^4}{24} \frac{k!}{(k+4)!}$$

Q5. $$\int_{x=0}^{5} \int_{y=10}^{20} f_{X,Y}(x,y)\, dy\, dx = 1$$

$$\Rightarrow \int_{x=0}^{5} \int_{y=10}^{20} ax(x+y)\, dy\, dx = 1$$

$$\int_{x=0}^{5} \int_{y=10}^{20} (ax^2 + axy)\, dy\, dx = 1$$

$$\int_{x=0}^{5} \left[ax^2 y + \frac{axy^2}{2}\right]_{10}^{20} dx = 1$$

$$\int_{x=0}^{5} \left(20ax^2 + \frac{400}{2}ax - 10ax^2 - \frac{100}{2}ax\right) dx = 1$$

$$\int_{x=0}^{5} (10ax^2 + 150ax)\, dx = 1$$

$$\left[\frac{10ax^3}{3} + \frac{150ax^2}{2}\right]_0^5 = 1$$

$$\frac{10a}{3} \times 5^3 + \frac{150a}{2} \times 5^2 = 1$$

$$\frac{1250a}{3} + 1875a = 1$$

$$\frac{6875}{3}a = 1$$

$$a = \frac{3}{6875}$$

~~$P\left(Y > \frac{1}{3}X + 10\right) = \int$~~

$$f_Y(y) = \int_{x=0}^{5} f_{X,Y}(x,y)\,dx$$

$$= \int_{x=0}^{5} (ax^2 + axy)\,dx$$

$$= \left[\frac{ax^3}{3} + \frac{ax^2 y}{2}\right]_0^5$$

$$= \frac{a \times 125}{3} + \frac{a}{2} \times 25y$$

$$= \frac{125}{3} \times \frac{3}{6875} + \frac{25}{2} \times \frac{3}{6875} y$$

$$= \frac{125}{6875} + \frac{75}{13750} y$$

$$= \frac{1}{55} + \frac{3}{550} y$$

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$$P\left(y > \frac{1}{3}x + 10\right) = \int_{y=\frac{1}{3}x+10}^{20} \left(\frac{1}{55} + \frac{3y}{550}\right) dy$$

$$= \left[\frac{1}{55}y + \frac{3y^2}{3 \times 550}\right]_{\frac{1}{3}x+10}^{20}$$

$$= \left(\frac{1}{55}y + \frac{y^2}{550}\right)_{\frac{1}{3}x+10}^{20}$$

$$= \frac{\frac{1}{3}x + 10}{55} + \frac{(\frac{1}{3}x + 10)^2}{550} - \frac{20}{55} - \frac{400}{550}$$

$$= \frac{x+30}{165} + \frac{(x+30)^2}{4950} - \frac{4}{11} - \frac{8}{11}$$

$$= \frac{x+30}{165} + \frac{x^2 + 900 + 60x}{4950} - \frac{12}{11}$$

$$= \frac{30x + 900 + x^2 + 900 + 60x}{4950} - \frac{12}{11}$$

$$= \frac{x^2 + 90x + 1800}{4950} - \frac{12}{11}$$

$$= \frac{x^2}{4950} + \frac{90x}{4950} + \frac{1800}{4950} - \frac{12}{11}$$

$$= \frac{x^2}{4950} + \frac{90x}{4950} + \frac{4}{11} - \frac{12}{11}$$

$$= \frac{x^2}{4950} + \frac{90x}{4950} - \frac{8}{11}$$

$$f_{Y/Y}(y) = \frac{f(x,y)}{f(x)} \Rightarrow f(x) = \frac{3}{6875}\int_{10}^{20} (x^2 + xy)\, dy$$

$$= \frac{3}{6875}\left(x^2 y + \frac{xy^2}{2}\right)_{10}^{20}$$

$$= \frac{3}{6875}\left(20x^2 + 200x - 10x^2 - 50x\right)$$

$$= \frac{3}{6875}\left(10x^2 + 150x\right)$$

$$= \frac{30x}{6875}(x + 15)$$

$$f_{Y/X} = \frac{3}{6875} \frac{x(x+y)}{30x} \frac{6875}{x+15}$$

$$= \frac{x+y}{10(x+15)}$$

$$\text{So } E(Y|X) = \int_{10}^{20} y \frac{(x+y)}{10(x+15)}\, dy$$

$$= \frac{1}{10(x+15)}\int_{10}^{20} (xy + y^2)\, dy$$

$$= \frac{1}{10(x+15)}\left(\frac{xy^2}{2} + \frac{y^3}{3}\right)_{10}^{20}$$

$$= \frac{1}{10(x+15)}\left(200x + \frac{8000}{3} - 50x - \frac{1000}{3}\right)$$

$$= \frac{1}{10(x+15)}\left(150x + \frac{7000}{3}\right)$$

$$= \frac{1}{x+15}\left(15x + \frac{700}{3}\right)$$

Q6. $X \sim Poi(\lambda)$

$\cancel{M_X(t) = e^{\lambda(e^{t}-1)}}$ $M_X(t) = e^{\lambda(e^t - 1)}$

$Y \sim Poi(\mu)$

$\cancel{M_Y(t) = e^{\mu(e^t-1)}}$

$M_Y(t) = e^{\mu(e^t - 1)}$

Let $Z = X - Y$

$$M_Z(t) = E(e^{tZ}) = E(e^{t(X-Y)})$$
$$= E(e^{tX} \cdot e^{-tY})$$
$$= \frac{E(e^{tX})}{E(e^{tY})}$$
$$= \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$$
$$= e^{(\lambda-\mu)(e^t-1)}$$

$\Rightarrow Z \sim Poi(\lambda - \mu)$

Let $Z = X + Y$

$$M_Z(t) = E(e^{tZ}) = E(e^{t(X+Y)})$$
$$= E(e^{tX} \cdot e^{tY})$$
$$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$$
$$= e^{(\mu+\lambda)(e^t-1)}$$

$\therefore Z \sim Poi(\lambda + \mu)$

Q7. i) $var(X|Y=2) = E(X^2|Y=2) - (E(X|Y=2))^2$

$$E(X^2|Y=2) = \frac{\Sigma x^2 . P(X=x, Y=2)}{P(Y=2)}$$

$$= \frac{1\times 0}{0.6} + \frac{4\times 0.3}{0.6} + \frac{9\times 0.1}{0.6} + \frac{16\times 0.2}{0.6}$$

$$= \underline{\underline{8.8333}}$$

ii) $$E(X|Y=2) = \frac{\Sigma x . P(X=x, Y=2)}{P(Y=2)}$$

$$= \frac{1\times 0}{0.6} + \frac{2\times 0.3}{0.6} + \frac{3\times 0.1}{0.6} + \frac{4\times 0.2}{0.6}$$

$$= \underline{\underline{2.8333}}$$

$$\therefore var(X|Y=2) = 8.8333 - (2.8333)^2$$

$$= \underline{\underline{0.80571111}}$$

ii) $$f(u,v) = \frac{48}{67}(2uv - u^2)$$

$$f_V(v) = \int_{u=0}^{1} \frac{48}{67}(2uv - u^2)\, du$$

$$= \frac{48}{67}\left(\frac{2u^2 v}{2} - \frac{u^3}{3}\right)_0^1$$

$$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right] = \frac{48}{67}\frac{(3v-1)}{3}$$

$$\cancel{= \frac{48}{67} \times \frac{2}{3}} = \underline{\underline{\frac{16}{67}(3v-1)}}$$

$$\frac{f(u,v)}{f(v)} = \frac{48}{67} \frac{(2uv - u^2)}{16(3v-1)} \times 67$$

$$= \frac{3}{3v-1}(2uv - u^2)$$

$$\text{Now, } E(U/V=v) = \frac{3}{3v-1}\int_0^1 u(2uv - u^2)\,du$$

$$= \frac{3}{3v-1}\int_0^1 (2u^2 v - u^3)\,du$$

$$= \frac{3}{3v-1}\left[\frac{2vu^3}{3} - \frac{u^4}{4}\right]_0^1$$

$$= \frac{3}{3v-1}\left(\frac{2v}{3} - \frac{1}{4}\right)$$

$$= \frac{3}{3v-1}\left(\frac{8v-3}{12}\right)$$

$$= \frac{8v-3}{4(3v-1)}$$

8) $X \sim \text{Gamma}(3, 2)$

$E(Y|X=x) = 3x+1$

$Var(Y|X=x) = 2x^2+5$

$$E[Y] = E[E(Y|X=x)] = E(3x+1)$$

$$= 3E[X] + 1$$

$$= \left(3 \times \frac{3}{2}\right) + 1$$

$$= \frac{11}{2}$$

$$V[Y] = E[V(Y|X=x)] + V[E[Y|X=x]]$$

$$= E(2x^2+5) + V(3x+1)$$

$$= 2E[X^2] + 5 + 9V[X].$$

$$V[X] = \frac{3}{4} = E[X^2] - (E[X])^2$$

$$\Rightarrow \frac{3}{4} = E[X^2] - \frac{9}{4}$$

$$\Rightarrow E[X^2] = \frac{12}{4} = 3$$

$$\therefore V[Y] = (2\times 3) + 5 + \left(9 \times \frac{3}{4}\right)$$

$$= 11 + \frac{27}{4}$$

$$= \underline{\underline{71/4}}$$

9) i) $\text{cov}(X,Z) = \text{cov}(X, X+Y)$

$$= \text{cov}(X,X) + \text{cov}(X,Y)$$

$$= \text{var}(X) + \text{cov}(X,Y)$$

$$\rho = -0.3 = \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}}$$

$$\Rightarrow -0.3 = \frac{\text{cov}(X,Y)}{2}$$

$$\Rightarrow \text{cov}(X,Y) = -0.6$$

$$\therefore \text{cov}(X,Z) = 4 - 0.6 = \underline{\underline{3.4}}$$

ii) $Var(Z) = var(X+Y)$

$\cancel{= var} = var(X) + var(Y) + 2cov(X,Y)$

$= 4 + 1 \cancel{+} - (0.6 \times 2)$

$= 5 - 1.2$

$= 3.8$

10) $$\int_{x=1}^{\infty}\int_{y=1}^{\infty} k x^{-\alpha} e^{-y/\beta}\, dy\, dx = 1$$

$$\Rightarrow \int_{x=1}^{\infty} k \int_{y=1}^{\infty} x^{-\alpha} e^{-y/\beta}\, dy\, dx = 1$$

$$\Rightarrow \int_{x=1}^{\infty} k x^{-\alpha} \int_{y=1}^{\infty} e^{-y/\beta}\, dy\, dx = 1$$

$$\Rightarrow \int_{x=1}^{\infty} k x^{-\alpha} \left[\frac{e^{-y/\beta}}{-y^2/2\beta}\right]_1^{\infty} dx = 1$$

$$\Rightarrow \int_{x=1}^{\infty} k x^{-\alpha} \left(\frac{e^{-\infty}}{-\infty} - \frac{e^{-1/\beta}}{-1/2\beta}\right) dx = 1$$

$$\Rightarrow \int_{x=1}^{\infty} k x^{-\alpha} \times -\frac{e^{-1/\beta}}{-1/2\beta}\, dx = 1$$

$$\Rightarrow \int_{x=1}^{\infty} k x^{-\alpha} \times 2\beta \times e^{-1/\beta}\, dx = 1$$

$$\Rightarrow k \times 2\beta \times e^{-1/\beta} \int_{x=1}^{\infty} x^{-\alpha}\, dx = 1$$

$$\Rightarrow 2\beta k e^{-1/\beta} \times \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^{\infty} = 1$$

$$\Rightarrow 2k\beta e^{-1/\beta} \times \left(\frac{x^{-\infty}}{-\infty} - \frac{1^{-\alpha+1}}{-\alpha+1} \right) = 1$$

$$\Rightarrow 2k\beta e^{-1/\beta} \times \frac{-1}{1-\alpha} = 1$$

$$\Rightarrow k = \frac{-(1-\alpha)}{2\beta e^{-1/\beta}}$$

$$\Rightarrow k = \frac{\alpha - 1}{2\beta e^{-1/\beta}}$$

$$\Rightarrow k = \frac{(\alpha-1) e^{1/\beta}}{2\beta}$$