FINANCIAL MATHEMATICS

1. The nominal rate of discount per annum convertibly quarterly is 8%.
i) Calculate the equivalent force of interest. δ
ii) Calculate the equivalent rate of interest per annum i
iii) Calculate the equivalent nominal rate of discount per annum convertible monthly.

→ $d^{(4)} = 8\%$

ii) $d^{(p)} = p\left[1-(1-d)^{1/p}\right]$

$d^{(4)} = 4\left[1-(1-d)^{1/4}\right]$

$\frac{0.08}{4} = 1-(1-d)^{1/4}$

$0.98 = (1-d)^{1/4}$

$\therefore d = 7.763184\%$

$d = \frac{i}{1+i}$ $\therefore 0.07763184 + 0.07763184i = i$

$\therefore 0.07763184 = i - 0.07763184i$

$\therefore i = 0.0841657$

$\therefore i = 8.41657\%$

i) $\delta = \ln(1+i)$

$\delta = \ln(1+0.0841657)$

$\delta = \ln(1.0841657)$

$\therefore \delta = 8.081082\%$

iii) $d^{(12)} = 12\left[1-(1-d)^{1/12}\right]$

$= 12\left[1-(1-0.07763184)^{1/12}\right]$

$= 0.08053933$

$\therefore d^{(12)} = 8.0539339 4\%$

2. The force of interest $\delta(t) = 0.004t + 0.0002t^2$ for all t

i) Find Present value of sum of rupees 1000 payable at the end of 10 years from now

ii) Find constant annual effective rate of interest over a 10 year period.

$\rightarrow$ Amount due $= C = 1000$ $\quad n = 10$

$$PV_0 = C \cdot v(0,10).$$

$$= 1000 \cdot e^{\int_0^{10} -0.004t + 0.0002t^2 \cdot dt}$$

$$= 1000 \cdot e^{-\left[\frac{0.004t^2}{2} + \frac{0.0002t^3}{3}\right]_0^{10}}$$

$$= 1000 \cdot e^{-\left[0.002(100) + \frac{0.0002 \times 1000}{3}\right]}$$

$$= 1000 \cdot e^{-0.266666}$$

$$= 765.9283$$

$$\therefore PV_0 = ₹\,765.9283$$

ii)

$$i = e^{\int_0^{10} (0.04t + 0.0002t^2)dt} - 1$$

$$= e^{(0.002 \times 100) + \left(\frac{0.0002 \times 1000}{3}\right)} - 1$$

$$= e^{(0.3 + 0.0667)} - 1$$

$$= 0.30565.$$

3. i) Explain the relationship $d = iv$ by general reasoning where d is the effective annual rate of discount and i is the effective annual rate of interest.

→ Let us assume that a person borrow 1 at an effective rate of discount d. then in effective, the original principal is $1-d$, and the amount of discount is d. However, from the basic defination of i as the ratio of the amount of discount to the principal, we obtain:-

$$i = \frac{d}{1-d}$$

This formula expresses i as a function of d

$$i = \frac{d}{1-d}$$

$$i - id = d$$

$$d(1+i) = i$$

$$d = \frac{i}{1+i}$$

∴ Discount factor,

$$v = \frac{1}{1+i}$$

$$\therefore d = i \times \frac{1}{1+i}$$

$$d = iv$$

ii) a) $\frac{d^{(4)}}{4} = 2.25\%$

$$d = 1 - \left(1 - \frac{d^{(4)}}{4}\right)^{4}$$

$$d = 1 - \left(1 - \frac{0.0225}{4}\right)^{4}$$

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$$\therefore d = 0.022311$$

$$d^{(2)} = 2\left[1-(1-0.022311)^{1/2}\right]$$

$$d^{(2)} = 2.244\ \%$$

b) $d^{(0.5)} = 5\%$

$$d = 1-\left(1-\frac{0.05}{0.5}\right)^{0.5}$$

$$d = 0.051317$$

$$\therefore d^{(2)} = 2\left[1-(1-0.051317)^{1/2}\right]$$

$$\therefore d^{(2)} = 5.19\%$$

4. ii) Given $i = 0.08$, Calculate

a) $d^{(12)} = ?$

$$d = \frac{i}{1+i} = \frac{0.08}{1.08} = 0.0740740740$$
$$= 7.40\ \%$$

$$\therefore d^{(p)} = p\left[1 - (1-d)^{1/p}\right]$$

$$d^{(12)} = 12\left[1 - (1 - 0.0740740740)^{1/12}\right]$$
$$d^{(12)} = 12\left[1 - 0.993607101988\right]$$
$$d^{(12)} = 12\left(0.006392898011\right)$$
$$d^{(12)} = 0.0767147761 4$$
$$\therefore d^{(12)} = 7.6714\ \%$$

b) $i^{(365)}$

$$i^{(p)} = p\left[(1+i)^{1/p} - 1\right]$$
$$\therefore i^{(365)} = 365\left[(1+0.08)^{1/365} - 1\right]$$
$$i^{(365)} = 365\left[1.000210874 - 1\right]$$
$$i^{(365)} = 365\left[0.00021087439\right]$$
$$i^{(365)} = 0.076969155$$
$$\therefore i^{(365)} = 7.6969155\ \%$$

c) $\delta = ?$

$$\delta = \ln(1+i)$$
$$\delta = \ln(1+0.08)$$
$$\delta = \ln(1.08)$$
$$\therefore \delta = 7.696104 11\ \%$$

d) $i^{(0.5)} = ?$

$$i^{(0.5)} = 0.5\left[(1+0.08)^{1/0.5} - 1\right]$$
$$i^{(0.5)} = 0.5\left[1.1664 - 1\right]$$
$$i^{(0.5)} = 0.5\left[0.1664\right]$$
$$i^{(0.5)} = 0.0832$$
$$\therefore i^{(0.5)} = 8.32\ \%$$

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5. i) The rate of discount per annum convertible quarterly is 6%. Calculate:

a) The equivalent rate of interest per annum convertible half yearly.

→ $d^{(4)} = 6\%$

$i^{(6)} = ?$

$d^{(P)} = P\left[1-(1+i)^{-1/P}\right]$

$\therefore d^{(4)} = 4\left[1-(1+i)^{-1/4}\right]$

$\frac{0.06}{4} = 1-(1+i)^{-1/4}$

$(1+i)^{-1/4} = 1-0.015$

$1+i = (0.985)^{-4}$

$i = 1-1.0623193153$

$\therefore i = 6.23193153\%$

$i^{(P)} = P\left[(1+i)^{1/P}-1\right]$

$\therefore i^{(6)} = 6\left[(1+0.0623193)^{1/6}-1\right]$

$i^{(6)} = 6\left[1.0101266899-1\right]$

$i^{(6)} = 0.0607601$

$\therefore i^{(6)} = 6.076\%$

b) The equivalent rate of discount per annum convertible monthly. $(d^{(12)} = ?)$

$\therefore d^{(12)} = 12\left[1-(1+i)^{-1/12}\right]$

$d^{(12)} = 12\left[1-(1+0.06231931)^{-1/12}\right]$

$d^{(12)} = 12\left[1-0.994974789\right]$

$d^{(12)} = 0.0603025$

$\therefore d^{(12)} = 6.03025\%$

ii) On 15th April, 2005 Amit borrowed Rs 2,00,000 to be repaid one year later by single payment of Rs. 2,20,000. Amit repaid the loan early on 17th July, 2005,

a) Find the sum paid by Amit to terminate the contract assuming that the interest is reduced proportionally for early settlement.

$i = 10\%$

P

2,00,000 — 2,20,000

→ Loan was paid in 93 days
So total sum paid within
93 days = 200000 + 20000 (93/365)
= 205095.89

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6. i) The manufacturer of a certain toy sells to retailers on either of the following terms:
a) Cash Payment : 30% below the recommended retail price
b) Six months credit : 25% below the recommended r.
Find the effective annual rate of discount offered by the manufacturer to the retailers who pay cash as opposed to those retailers who accept the credit terms.

Let us assume cash payment of 100

$\therefore$ Discount value of 100 at time 0 $= 100 \times \frac{30}{70}$

$= 70.$

At six month $= 100 \times 25\% = 75$

$\therefore PV_0 = AV(1-d)^n$

$70 = 75(1-d)^{1/2}$

$\left(\frac{70}{75}\right)^2 = (1-d)$

$\therefore d = 0.128889$

$\therefore d = 12.88\%$

$d^{(12)} = 12\left[1-(1-0.128889)^{1/12}\right]$

$= 13.71956\%$

$i^{(2)} = ?$

$i = \frac{0.1288889}{1-0.128889} = 0.1479593$

$i^{(12)} = 2\left[(1+0.1479593)^{1/12} - 1\right]$

$i^{(12)} = 14.2857\%$

$\delta = \ln(1-d)$

$= 13.7986\%$

7. If an investment fund offers to increase INR 20,000 to INR 26,000 in 17 months, calculate the following:

i) The nominal rate of interest per annum convertibly quarterly.

→ $PV_0 = 20000$

$AV_{19} = 26000$ $\quad 4 \times \frac{12}{12}$

So, $20000 = 26000 \left(1 + \frac{i^{(4)}}{4}\right)$

$\therefore \left(\frac{20000}{26000}\right) = 1 + \frac{i^{(4)}}{4}$

$\therefore i^{(4)} = 18.96\ \%$

ii) $d^{(2)} = ?$

$i = \left(1 + \frac{0.1296}{4}\right)^4 - 1$

$i = 20.34541$

$d = \frac{i}{1+i} = \frac{0.2034541}{1+0.2034541} = 16.90463$

$\therefore d^{(12)} = 2\left[1-(1-0.1690463)^{1/2}\right]$

$d^{(2)} = 16.69\%$

iii) $26{,}000 = 20000\,(1+i)^{18/12}$

$\left(\dfrac{26{,}000}{20{,}000}\right)^{12/18} = 1+i$

$\therefore i = 21.15\%$

8. Explain with the help of graph, for an effective interest rate of 8% p.a, the relationship between equivalent nominal rates of interest convertible pthly $(i^{(p)})$ and p.

$i = 8\%$

$i^{(20)} = 7.\cancel{7…}\%$

$i^{(30)} = 7.705\%$

$i^{(40)} = 7.703\%$

$i^{(50)} = 7.702\%$

$i^{(60)} = 7.701\%$

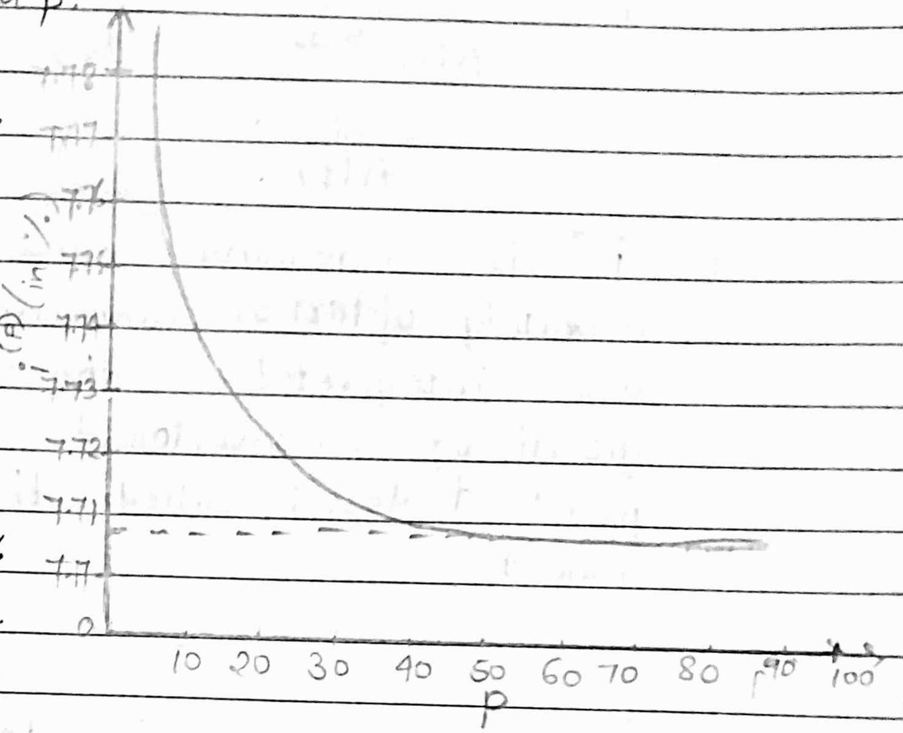
$i^{(70)} = 7.700\%$

$i^{(80)} = 7.769\%$

$i^{(85)} = 7.707\%$

$i^{(90)} = 7.768\%$

$i^{(100)} = 7.699\%$



Now, for this as we can see as the time(p) increases the equivalont nominal rates of interest decreases. It can also be seen in table that the more frequently compounding takes place (i.e. p increases), the smaller is the equivalent nominal annual rate. The change is less significant, however, in going from years to years, so we see that it is less significant. If p is increased, the time interval $[t, t+\frac{1}{p}]$ decreases, and we are focusing more and more closely on the investment performance. Taking the limit of $i^{(p)}$ as $p \to \infty$ results in

$$i^{(\infty)} = \lim_{p \to \infty} i^{(p)} = \lim_{p \to \infty} p \times \frac{A\left(t+\frac{1}{p}\right) - A(t)}{A(t)}$$

The limit can be reformulated by making the following variable susbstitution. Define the variable

h to be $h = \frac{1}{p}$, so that $h \to 0$ as $p \to \infty$. The limit can then be written in the form

$$i^{(\infty)} = \frac{1}{A(t)} \cdot \lim_{h \to 0} \frac{A(t+h) - A(t)}{h} = \frac{1}{A(t)} \cdot \frac{d}{dt} A(t)$$

$$= \frac{A'(t)}{A(t)}$$

$i^{(\infty)}$ is a nominal annual interest rate compounded infinitely often or compounded continuously. $i^{(\infty)}$ is also interpreted as the instataneous rate of growth of the investment per dollar invested at time point t and is called the force of interest at time t.

Q.9. i) Define force of interest.

→ δ is the nominal rate of interest per unit time convertible continuously (or momently). This is also referred to as the rate continuously compounded. We call it the force of interest.

ii) If $i^{(2)} = 0.0775$

Calculate i, δ, $d^{(12)}$, $i^{(1/2)}$

→ $i^{(2)} = 0.0775$

$\therefore i^{(p)} = p\left[(1+i)^{1/p} - 1\right]$

$i^{(2)} = 2\left[(1+i)^{1/2} - 1\right]$

$\frac{0.0775}{2} = (1+i)^{1/2} - 1$

$0.03875 + 1 = (1+i)^{1/2}$

$(1.03875)^2 = 1+i$

$\frac{1}{2}$ 0.5

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2d

$1.0790015625 - 1 = i$

$\therefore \quad i = 7.9001\ \%$

Now, $\delta = \ln(1+i)$

$\delta = \ln(1.0790015625)$

$\delta = 0.0760361$

$\therefore \delta = 7.6036\ \%$

Now,

$d^{(p)} = p\left[1-(1+i)^{-1/p}\right]$

$d^{(12)} = 12\left[1-(1+0.0790015625)^{-1/12}\right]$

$d^{(12)} = 12\left[1-0.99368368776\right]$

$d^{(12)} = 12(0.006316631223)$

$d^{(12)} = 0.075795746$

$\therefore d^{(12)} = 7.5795\%$

Now,

$i^{(1/2)} = \frac{1}{2}\left[(1+i)^{1/2}-1\right]$

$i^{(1/2)} = 1/2\left[(1+0.0790015625)^{2}-1\right]$

$i^{(1/2)} = 1/2\,(0.16424443718)$

$i^{(1/2)} = 0.082122185$

$\therefore i^{(1/2)} = 8.21221\%$

10. $\delta(t)$ the force of interest per annum at time t years is defined as :-

$$\delta(t) = \begin{cases} 0.08 \text{ for } 0 \le t \le 5 \\ 0.06 \text{ for } 5 \le t \le 10 \\ 0.04 \text{ for } t \ge 10 \end{cases}$$

Using $\delta(t)$, Derive the expression for $v(t)$, the present value of 1 due at time t.

→ When, $0 \le t \le 5t$

$$v(t) = e^{-\int_0 0.08}$$

$$= e^{-0.08t}$$

When, $5 \le t \le 10$

$$v(t) = e^{-\int_0^5 0.08 - \int_5^t 0.06}$$

$$= e^{{}_5^0[0.08t] + {}_t^5[0.06t]}$$

$$= e^{-0.4+0.3-0.06}$$

$$= e^{-0.06t - 0.1}$$

When, $t \geq 10$

$$v(t) = e^{-\int_0^5 0.08 - \int_5^{10} 0.06 - \int_{10}^t 0.04}$$

$$= e^{{}_5^0[0.08t] + {}_{10}^5[0.06t] + {}_t^{10}[0.04t]}$$

$$= e^{-0.4+0.3-0.6+0.4-0.04t}$$

$$= e^{-0.04t-0.3}$$

11. a) A 9-month loan, repayable by a single payment of 50,000 is issued at a rate of commercial discount of 18% p.a. What amt was initially lent to the borrower?

→ $$PY = 50000\left[\frac{1}{1-0.18}\right]^{9/12}$$

$$= 43085.426$$

b) (i) $\delta = 6\%$.

$\therefore d = 1 - e^{-\delta}$

$\therefore d = 5.82354\%$.

$d^{(12)} = 12\left[1-(1-5.82354\%)^{1/12}\right]$

$\therefore d^{(12)} = 5.985018\%$

(ii) $d^{(4)} = 6\%$.

b) $d = \left[1-\left(1-\frac{d^{(4)}}{4}\right)^{4}\right]$

$d = \left[1-(1-0.015)^{4}\right]$

$\therefore d = 5.866346\%$.

$i = \frac{d}{1-d} = 6.2319316\%$.

$i^{(4)} = 4\left[(1+0.062319316)^{1/4} - 1\right]$

$\therefore i^{(4)} = 6.09137\%$.

12. $PV_0 = 7049$, $i^{(12)} = 6\%$, $i^{(4)} = 1.5\%$, $d^{(12)} = 6\%$.

$d = 1-\left(1-\frac{d^{(12)}}{12}\right)^{12}_{12\times 2} = 0.0584$

$AV_6 = 7049\left[\left(1+\frac{i^{(12)}}{12}\right) \cdot \left(1+\frac{i^{(4)}}{4}\right)^{4\times\frac{10}{12}} \cdot (1-d)^{10/12}\right]$

$AV_6 = 7049\left[\left(1+\frac{0.06}{12}\right)^{24} \cdot \left(\frac{1+0.015}{4}\right)^{10} \cdot (1-0.0884)\right]$

$AV_6 = 7049\left[(1.127150)\cdot(1.03814)(0.913692)\right]$

$AV_6 = 7536.4154$.

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13. i) $(1+i)^n = 2$

$(1-1)^n = 2$

$\therefore n = 7.3$ years

ii) $d^{(12)} = 10\%$

$d = 1 - \left[1 - \frac{d^{(12)}}{12} \right]^{12}$

$d = 9.55\%$

$\therefore \frac{1}{2} = (1-d)^n$

$\frac{\ln(0.5)}{\ln(0.9045)} = n$

$\therefore n = 6.9$ years