PROBABILITY AND STATISTICS ASSIGNMENT - 1

1) Let X be the amount of a home insurance claim and Y the amount of a car insurance claim.

$X \sim N(800, 100^2)$ $\qquad$ $Y \sim N(1200, 300^2)$

we require:-

$$P[(Y_1 + Y_2 + Y_3) > (X_1 + X_2 + X_3 + X_4) + 800]$$

$$= P[(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800]$$

We need distribution of $(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4)$

$$= \sim N(3 \times 1200 - 4 \times 800,\ 3 \times 300^2 + 4 \times 100^2)$$

$$\sim N(400, 3,10,000)$$

$$= P\left(Z > \left(\frac{800 - 400}{\sqrt{3,10,000}}\right)\right)$$

$$= P(Z > 0.718)$$

$$= 1 - P(Z < 0.718) = 0.236.$$

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4) Probability, $P(X=n) = \frac{1}{n!} G^{(n)}_X(0)$

$\Rightarrow P(V=4) = \frac{1}{4!} G^{4}_V(0)$

finding $G^{(4)}_V(t)$

$G'V(t) = kp^k q(1-qx)^{-k-1}$

$G''V(t) = -(-k-1)kp^k q^2(1-qx)^{-k-2}$

$G'''V(t) = (-k-2)(-k-1)kp^k q^3(1-qx)^{-k-3}$

$G^4 V(t) = -(-k-3)(-k-2)(-k-1)kp^k q^4(1-q^n)$

$G^{(4)}_V(0) = -(-k-3)(-k-2)(-k-1)kp^x q^4.$

3.)

(i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$, $x \geq \alpha$ & $\lambda, \alpha > 0.$

using the fuction of an MGF:

$M_x(t) = E(e^{tx}] = \int_\alpha^\infty e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$

$= \int_\alpha^\infty \lambda e^{\lambda x} e^{-(\lambda-t)x} dx$

$= \left[\frac{-\lambda e^{\lambda x}}{\lambda - t} e^{-(\lambda-t)x}\right]_\alpha^\infty$

$$= \frac{\lambda}{\lambda - t} e^{(t\alpha)} \qquad \text{provided } t < \lambda$$

(ii) $$M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$M'_x(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-2} e^{t\alpha} +$$

$$\alpha\left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$\Rightarrow E(x) = M'_x(0) = \frac{1}{\lambda} + \alpha$$

$$M''_x(t) = \frac{2}{\lambda^2}\left(1 - \frac{t}{\lambda}\right)^{-3} e^{tx} + \frac{2\alpha}{\lambda}\left(1 - \frac{t}{\lambda}\right) +$$

$$\alpha^2\left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$\Rightarrow E(x^2) = M''_x(0) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$\text{var}(x) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left(\frac{1}{\lambda} + \alpha\right)$$

$$\text{var}(x) = \frac{1}{\lambda^2}$$

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5) $f(x,y) = ax(x+y) \qquad 0 < x < 5,$

we know that, $\qquad 10 < y < 20$

$$\int_{10}^{20}\int_{0}^{5} ax(x+y)\, dx\, dy = 1$$

$$= \int_{10}^{20}\int_{0}^{5} a\,(x^2 + xy)\, dx\, dy = 1$$

$$= \int_{10}^{20} a\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5 dy = 1$$

$$= \int_{10}^{20} a\left[\frac{125}{3} + \frac{25}{2}y\right] dy = 1$$

$$= a\left[\frac{125}{3}y + \frac{25}{4}y^2\right]_{10}^{20} = 1$$

$$= a\left[\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right] = 1$$

$$= a\,(2291.666) = 1$$

$$\therefore\ a = \underline{0.00044}$$

$$E\left(Y/x\right) = \int_y y \left(\frac{(x,y)}{f(x)} \right) dy$$

$$f(x) = \int f(x,y)\,dy$$

$$= \int_{10}^{20} a\left(x^2 + xy\right) dy$$

$$= a\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= a\left(20x^2 + 200x - 10x^2 - 50x\right]$$

$$= a\left[10x^2 + 150x\right]$$

$$E\left(Y/x\right) = \int_{10}^{20} y \; \frac{a\left[x^2 + xy\right]}{a\left[10x^2 + 150x\right]} \, dy$$

$$= \int_{10}^{20} y \frac{(x+y)}{(10x + 150)} \, dy$$

$$= \frac{1}{10x + 150} \left[\frac{y^2}{2} x + \frac{y^3}{3} \right]_{10}^{20}$$

$$= \frac{1}{10x + 150} \left(150x + \frac{7000}{3} \right)$$

6.) Examining Arthur's Statments:-

$X \sim Poi(\lambda)$ & $Y \sim Poi(\mu)$

X has MGF

$$M_x(t) = e^{\lambda(e^t - 1)}$$

and

Y has MGF

$$M_y(t) = e^{\mu(e^t - 1)}$$

So the difference

X−Y has MGF

$$M_x(t)\, M_{-y}(t) = e^{\lambda(e^t-1)}\, e^{-\mu(e^t-1)}$$

$$= e^{(\lambda-\mu)(e^t-1)}$$

⇒ which is the moment generating function of a Poison with parameter $\lambda-\mu$

⇒ $X - Y \sim Poi(\lambda-\mu)$

So Arthur's Statment is true

Similarly examining christine's Statment:-

$X \sim Poi(\lambda)$ & $Y \sim Poi(\mu)$

$M_x(t) = e^{\lambda(e^t-1)}$ & $M_y(t) = e^{\mu(e^t-1)}$

So the sum $X+Y$ has MGF:

$$M_x(t)\, M_y(t) = e^{\lambda(e^t-1)}\, e^{\mu(e^t-1)}$$

$$= e^{(\lambda+\mu)(e^t-1)}$$

$\Rightarrow$ MGF of Poison with parameter $\lambda+\mu$

$\Rightarrow X+Y \sim Poi(\lambda+\mu)$

So christine's statment is also true.

10.) $kx^{-\alpha} e^{y/\beta}$ $\qquad 1 < x < \infty$

$1 < y < \infty$

Since the pdf must integerate to 1

$$\int_{y=1}^{\infty}\int_{x=1}^{\infty} kx^{-\alpha} e^{y/\beta}\, dx\, dy = 1$$

integerating over x values gives:-

$$\int_1^{\infty} kx^{-\alpha} e^{-y/\beta} = ke^{-y/\beta}\left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^{\infty}$$

$$= \frac{ke^{-y/\beta}}{\alpha-1}$$

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integrating this one's y values:-

$$\int_1^\infty \frac{k e^{-y/\beta}}{\alpha - 1} = \frac{k}{\alpha - 1}\left[\beta e^{-y/\beta}\right]_1^\infty$$

$$= \frac{k\beta e^{-1/\beta}}{\alpha - 1} = 1$$

$$\boxed{k = \frac{(\alpha - 1)e^{1/\beta}}{\beta}}$$

9.) $E[X] = 3$
$Sdx = 2$
$E(Y) = 4$
$Sdy = 1$

Correlation coefficient = -0.3
$z = x + y$

(i.) $cov(x, z) = cov(x, x+y)$
$= cov(x, x) + cov(x, y)$

$= var(x) + cov(x, y)$

Using correlation coefficient between X & Y gives:-

$$\text{corr}(x, y) = -0.3 = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x)\,\text{var}(y)}}$$

$$= \frac{\text{cov}(x, y)}{\sqrt{4 \times 1}}$$

$$\text{cov}(x, y) = -0.6$$

$$\Rightarrow \text{cov}(x, z) = 4 - 0.6 = 3.4$$

(iii) $\text{var}(z) = \text{cov}(z, z)$

$$\text{var}(z) = \text{cov}(x+y, x+y)$$

$$= \text{cov}(x, x) + 2\,\text{cov}(x, y) + \text{cov}(y, y)$$

$$= \text{var}(x) + 2\,\text{cov}(x, y) + \text{var}(y)$$

$$= 4 + 2(-0.6) + 1$$

$$= \underline{\underline{3.8}}$$

8) $X \sim \text{Gamma}(3, 2)$

$$E(Y/_{X=x}) = 3x + 1$$

$$\text{var}(Y/_{x} = x) = 2x^2 + 5$$

$$E(Y) = E(E(Y/_{X})) = E(3x + 1)$$

$$= 3E(x) + 1$$

But $E(x) = \frac{3}{2} = 1.5$

$\Rightarrow$ $E(Y) = 3(1.5) + 1$

$= 4.5 + 1$

$= \underline{\underline{5.5}}$

using $\text{var}(Y) = E[\text{var}(Y/_{x})] + \text{var}(E(Y/_{x}))$

$= \text{var}(Y) = E[2x^2 + 5] + \text{var}(3x + 1)$

$= 2E(x^2) + 5 + 9\,\text{var}(x)$

$= 2\left[(1.5)^2 + \left(\frac{9}{16}\right)\right] + 9\left(\frac{3}{4}\right)$

$= 2\,[2.8125) + 6.75$

$$\boxed{\text{var}(Y) = 12.375}$$

$$X$$

7.		1	2	3	4
	1	0.2	0	0.05	0.15
Y	2	0	0.3	0.1	0.2

(i) $E(X/Y=2) = 1 \times \frac{0}{0.6} + 2 \times \frac{0.3}{0.6} + 3 \times \frac{0.1}{0.6} + 4 \times \frac{0.2}{0.6}$

$= 0 + 1 + 1/2 + 4/3$

$= 1 + \frac{1}{2} + \frac{4}{3}$

$= \frac{6+3+8}{6}$

$\therefore E(X/Y=2) = \frac{17}{6}$.

$E(X^2/Y=2) = 1^2 \times \frac{0}{0.6} + 2^2 \times \frac{0.3}{0.6} + 3^2 \times \frac{0.1}{0.6} + 4^2 \times \frac{0.2}{0.6}$

$= \frac{53}{6}$

$\therefore var(X/Y=2) = E(X^2/Y=2) - [E(X/Y=2)]^2$

$= \frac{53}{6} - \left(\frac{17}{6}\right)^2$

$\therefore \; Var(X/Y=2) = 0.80\overline{55}$

(ii) $f(U,V) = \frac{48}{67}(2uv - u^2)$ $\quad 0<u<1$, $\; u/2 < v < 2$

$E(U/V=v) = \int u \cdot f(u/v)\, du$

$f(v) = \int_{u=0}^{1} \frac{48}{67}(2uv - u^2)\, du$

$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$

$= \frac{48}{67}\left[v - \frac{1}{3}\right]$

$\therefore \; f(u/v) = \frac{f(u,v)}{f(v)}$

$= \frac{\cancel{\frac{48}{67}}(2uv - u^2)}{\cancel{\frac{48}{67}}(v - \frac{1}{3})} = \frac{2uv - u^2}{v - 1/3}$

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$\therefore E\left(\cancel{x} u / V = v\right) = \int_0^1 u \cdot f(u/v)\, du$

$= \int_0^1 \cdot \frac{2u^2 v - u^3}{(v - 1/3)}\, du$

$= \left[\frac{\frac{2}{3} u^3 v - \frac{u^4}{4}}{v - 1/3} \right]_0^1$

$\therefore E\left(U / V = v\right) = \frac{8u^3 v - 3u^4}{4(3v - 1)}$