Calculus Assignment - 2

1) $\int_{\frac{1}{50}}^{2} \frac{e^{2/w}}{w^2} \cdot dw$

$\frac{2}{w} = t$; diff wrt to w

$\frac{2.(-1)}{w^2} = \frac{dt}{dw}$, $w = \frac{1}{50}$, ~~t~~ $t = 100$

$\frac{dw}{w^2} = -\frac{1}{2} dt$, $w = 2 = t = 1$

$= \frac{-1}{2}\int_{100}^{1} e^{t} \cdot dt = -\frac{1}{2}\left[e^{t}\right]_{100}^{1}$

$= -\frac{1}{2}\left[e - e^{100}\right] = \cancel{e^{100}} \frac{e(e^{99}-1)}{2}$

2) $\int \frac{2t^3+1}{(t^4+2t)^3} \cdot dt$; $t^4 + 2t = x$

diff wrt ~~x~~ t

$\cancel{2}\, 4t^3 + 2 = \cancel{\frac{dt}{dx}} \frac{dx}{dt}$

~~$2\,dx = dt$~~ $(2t^3+1) \cdot dt = \frac{dx}{2}$

~~$2[2t^3+1]$~~

$= \frac{1}{2}\int \frac{dx}{x^3} = \frac{1}{2}\int x^{-3} \cdot dt$

$= \frac{1}{2}\left[\frac{x^{-2}}{-2}\right] = \frac{-1}{4\cancel{8} x^2}$

$= \cancel{\frac{-1}{2t^4+4t}} = -\frac{1}{4}\left[\frac{1}{(t^4+2t)^2}\right] + C$

3) $f'(x) = x^4 + 3x - 9$ Int. both sides

$$\int f'(x) = \int x^4 + 3x - 9$$

$$\boxed{f(x) = \frac{x^5}{5} + \frac{3x^2}{2} - 9x + C}$$

4) $f(x) = \begin{cases} 6, & x > 1 \\ 3x^2, & x \le 1 \end{cases}$

$$\int_{-2}^{3} f(x).dx = \int_{-2}^{1} 3x^2.dx + \int_{1}^{3} 6.dx$$

$$= \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_{-2}^{1} + 6[x]_1^3$$

$$= \cancel{3}\left[1 - (-2)^3\right] + 6[3 - 1]$$

$$= 1 + 8 + 12 = 21$$

5) $\int 3(8y-1)e^{4y^2-y}.dy$

$4y^2 - y = t$

diff wrt y

$8y - 1 = \frac{dt}{dy} \Rightarrow dt = (8y-1).dy$

$$= 3\int e^t . dt = 3e^t$$

$$\Rightarrow 3e^{4y^2-y} + C$$

6) $f''(x) = 15\sqrt{x} + 5x^3 + 6$

$\xrightarrow{}$ Int. both sides

$$\int f''(x) = \int 15\sqrt{x} + 5x^3 + 6 . dx$$

$$f'(x) = \frac{15x^{3/2}}{\frac{3}{2}} + \frac{5x^4}{4} + 6x + C$$

$\xrightarrow{}$ Int. both sides

$$\int f'(x) = \int 10x^{3/2} + \frac{5x^4}{4} + 6x + C$$

$$f(x) = \frac{10x^{5/2}}{\frac{5}{2}} + \frac{5x^5}{4 \times 5} + \frac{6x^2}{2} + Cx + k$$

$$f(x) = 4x^{5/2} + \frac{x^5}{4} + 3x^2 + Cx + k$$

$$f(1) = 4 + \frac{1}{4} + 3 + C + k = -\frac{5}{4}$$

$$C + k = -\frac{6}{4} - 7$$

$$C + k = -\frac{34}{4} = -\frac{17}{2} \quad -i)$$

$$f(4) = 4 \times (4)^{5/2} + \frac{(4)^5}{4} + 3 \times 4^2 + 4C + k = 404$$

$$4 \times 2^5 + 4^4 + 3 \times 16 + 4C + k = 404$$

$128 + 256 + 48 + 4c + k = 404$

$4c + k = 404 - 432$

$4c + k = -28$ — ii)

$-c - k = 17/2$

$3c = \frac{17 - 56}{2}$

$c = \frac{-39}{6}$, $k = -17/2 + 39/6$

$\Rightarrow \frac{39 - 51}{6} \Rightarrow k = -2$

$$\Rightarrow f(x) = 4x^2\sqrt{x} + \frac{x^5}{4} + 3x^2 - \frac{39}{6}x - 2$$

7) $f(x+iy) + g(x-iy) = z$ — i)

$\longrightarrow$ Partially diff i) wrt x.

$P = \frac{\partial z}{\partial x} = f'(x+iy) + g'(x-iy)$ — a)

$\longrightarrow$ Partially diff i) wrt y

$Q = \frac{\partial z}{\partial y} = if'(x+iy) + (-i)g'(x-iy)$ — b)

$\rightarrow$ Partially diff a) wrt x

$R = \frac{\partial^2 z}{\partial x^2} = f''(x+iy) + g''(x-iy)$

$\longrightarrow$ Partially diff b) wrt y

$T = \frac{\partial^2 z}{\partial y^2} = i^2 f''(x+iy) + (-i)^2 g''(x-iy)$

$T = i^2 [f''(x+iy) + g''(x-iy)] = i^2 R \Rightarrow T = (\sqrt{-1})^2 R$

$T = -R$

or $\boxed{T + R = 0}$

8) $\frac{dr}{d\theta} = \frac{r^2}{\theta}$; $r(1) = 2$

$$\frac{dr}{r^2} = \frac{d\theta}{\theta}$$

Int both sides

$$\int \frac{dr}{r^2} = \int \frac{d\theta}{\theta}$$

$$\frac{r^{-1}}{-1} = \log\theta + C$$

$$\log\theta + \frac{1}{r} + C = 0$$

$$r(1) = 2$$

$$\log(1) + \frac{1}{2} + C = 0$$

$$C = -\frac{1}{2}$$

$$\boxed{\log\theta + \frac{1}{r} = \frac{1}{2}}$$

9) $\frac{dv}{dt} = 9.8 - 0.196v$

$\frac{dv}{dt} + 0.196v = 9.8$

~~P = 0.16~~

~~I.F~~ => this is of the form $\frac{dy}{dx} + Py = Q$

$P = 0.196,\ Q = 9.8$

$I.F = e^{\int P.dt} = e^{\int 0.196 dt} = e^{0.196t}$

$v.(IF) = \int (Q.\ IF).dt + C$

$ve^{0.196t} = \int 9.8.e^{0.196t}.dt$

$ve^{0.196t} = \frac{9.8e^{0.196t}}{0.196} + C$

$\Rightarrow 0.196ve^{0.196t} = 9.8e^{0.196t} = C$

$e^{0.196t}\left[0.196v - 9.8\right] = C$

10) $ty' + 2y = t^2 - t + 1$, $y(1) = 1/2$

$\cancel{t\frac{dy}{dt}}\ \frac{dy}{dt} + \frac{2y}{t} = t - 1 + t^{-1}$

$P = 2/t, \quad Q = t - 1 + t^{-1}$

$I.F = e^{\int P.dx} = e^{\int \frac{2}{t}.dt} = e^{2\log t}$

$y.e^{2\log t} = \int e^{2\ln t}.(t-1+t^{-1}).dt$

$y.e^{\ln t^2} = \int e^{\ln t^2}(t-1+t^{-1}).dt$

$yt^2 = \int t^2(t-1+t^{-1}).dt$

$y = \frac{1}{t^2}\int \left[t^3 - t^2 + t\right].dt$

$y = t^{-2}\left[\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2}\right]$

$y = \frac{t^2}{2} - \frac{t}{3} + \frac{1}{2} + C \quad (y(1) = 1/2)$

$\cancel{\frac{1}{2} = \frac{1}{2} - \frac{1}{3} + \frac{1}{2} + c}$ $\quad \cancel{\frac{1}{2}} = \cancel{\frac{1}{2}} - \frac{1}{3} + \frac{1}{2} + C$

$C = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$\Rightarrow 6y = 3t^2 - 2t + 3 - 1$

$\Rightarrow \boxed{6y = 3t^2 - 2t + 2}$

11) $\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$, $y(0) = -1$

$\frac{dy}{y^3} = \frac{x}{\sqrt{1+x^2}} \cdot dx$ $\longrightarrow$ $\left[\begin{array}{l} 1+x^2 = t, \quad 2x\,dx = dt \\ \quad x\,dx = \frac{1}{2}dt \\ = \frac{1}{2}\int \frac{dt}{t^{1/2}} = \frac{1}{2}\frac{t^{1/2}}{(1/2)} \\ = (1+x^2)^{1/2} \end{array} \right]$

Int. both sides

$\int \frac{dy}{y^3} = \int \frac{x}{\sqrt{1+x^2}} \cdot dx$

$\frac{-1}{2y^2} = (1+x^2)^{1/2} + C$

$y(0) = 1$

$-\frac{1}{2} = (1+0)^{1/2} + C$

$C = -\frac{1}{2} - 1 = \boxed{-\frac{3}{2} = C}$

$\Rightarrow -\frac{1}{2y^2} = (1+x^2)^{1/2} - \frac{3}{2}$

$\Rightarrow 2(1+x^2)^{1/2} + y^{-2} = 3$

$\Rightarrow 2y^2(1+x^2)^{1/2} + 1 = 3y^2$

$\Rightarrow 2y^2(1+x^2)^{1/2} - 3y^2 + 1 = 0$

$\sqrt{1+x^2}$ is continuous for all pts. The interval of validity must include the value $x = 0$ as well, which is mention in the question

$\sqrt{1+x^2}$ is continuous for all real values of x

Thus interval of validity $= (-\infty, \infty)$

12) $f(x) = x^3 - 10x^2 + 6$ for $x = 3$

Taylor Series $\rightarrow f(x) = f(h) + \frac{(x-h)}{1!} f'(h) + \frac{(x-h)^2}{2!} f''(h) + \ldots$

$f(3) = -57$,

$f'(x) = 3x^2 - 20x \;;\; f'(3) = -33$

$f''(x) = 6x - 20 \;;\; f''(3) = -2$

$f'''(x) = 6 \;;\; f'''(3) = 6$

$f''''(x) = 0$

$$f(x) = -57 + (x-3) \times (-33) + \frac{(x-3)^2}{2} \times (-2) + \frac{(x-3)^3}{6} \times 6$$

$$= -57 - 33(x-3) - (x-3)^2 + (x-3)^3$$

3) $f(x) = (1+x)^u$

Maclaurian Series $\rightarrow f(x) = f(0) + \frac{x}{1!} \cdot f'(0) + \frac{x^2}{2!} f''(0) + \ldots$

$f(0) = 1^u = 1$

$f'(x) = u(1+x)^{u-1} \;;\; f'(0) = \cancel{1 \times \log u = \log} u \; u$

$f''(x) = \cancel{\log u . (1+x)} \; u(u-1)(1+x)^{u-2}, \; f''(0) = u.(u-1)$

$f'''(x) = u(u-1)(u-2)(1+x)^{u-3}, \; f'''(0) = u.(u-1)(u-2)$

$$f(x) = 1 + xu + \frac{x^2}{2}(u)(1-u) + \frac{x^3}{6}(u)(u-1)(u-2) + \dots \Big]$$

14) $f(x) = \sqrt{1+x}$

$$f(0) = \sqrt{1} = 1$$

$$f'(x) = \frac{1}{2\sqrt{1+x}}, \quad f'(0) = \frac{1}{2}$$

$$f''(x) = \cancel{-\frac{x}{4\sqrt{1+x}}} - \frac{1}{4(1+x)^{3/2}}, \quad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8(1+x)^{5/2}}, \quad f'''(0) = \frac{3}{8}$$

$$f(x) = 1 + \frac{x}{2} - \frac{1}{4}\left(\frac{x^2}{2}\right) + \frac{3}{8}\left(\frac{x^3}{6}\right) + \dots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$$

5) $f(x) = e^{-6x}$ about $x = -4$

$$f(k) = f(-4) = e^{24}$$

$$f'(x) = -6e^{-6x}, \quad f'(\overset{-4}{\cancel{k}}) = -6e^{24}$$

$$f''(x) = 36e^{-6x}, \quad f''(\overset{-4}{\cancel{k}}) = 36e^{24}$$

$$f'''(x) = 216e^{-6x}, \quad f'''(\overset{-4}{\cancel{k}}) = -216e^{24}$$

$$f(x) = e^{24} + (x+4)(-6e^{24}) + \frac{(x+4)^2}{2}(36e^{24}) + \frac{(x+4)^3}{6}(-216e^{24}) + \dots$$

$$f(x) = e^{24}\left[1 - 6(x+4) + 18(x+4)^2 - 36(x+4)^3 \ldots\right]$$

6) $f(x) = \ln(3+4x)$ about $x = 0$

$f(k) = f(0) = \ln(3)$

$f'(x) = \frac{1}{3+4x} \cdot (4)$; $f'(0) = \frac{4}{3}$

$f''(x) = \frac{-4 \times 4}{(3+4x)^2}$; $f''(0) = -\left(\frac{4}{3}\right)^2$

$f'''(x) = \frac{+16 \times 2 \cancel{(3+4x)} \cdot (4)}{(3+4x)^3}$; $f'''(0) = \frac{4^3}{3^4} \times 6 = \left(\frac{4}{3}\right)^3 \times 2$

$= \frac{4^3 \times 2}{(3+4x)^3}$

$f(x) = \ln(3) + \frac{4}{3}x - \left(\frac{4}{3}x\right)^2 \times \frac{1}{2} + \left(\frac{4}{3}x\right)^3 \times \frac{2}{6} \ldots$

$= \ln(3) + \frac{4}{3}x - \frac{1}{2}\left(\frac{4x}{3}\right)^2 + \frac{1}{3}\left(\frac{4x}{3}\right)^3 + \ldots$