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Probability & Statistics Assignment-2

1. Let x_i be the claim size of an home insurance
y_i be the claim size of an car insurance

So, ATQ $X \sim N(800, 100^2)$
$Y \sim N(1200, 300^2)$

$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$

$P(Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800)$

$(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim N(3 \times 1200 - 4(800), 300^2(3) + 100^2)$
$\sim N(400, 310000)$

$P(Z > \frac{800 - 400}{\sqrt{310000}})$

$P(Z > 0.71842) = 1 - P(Z < 0.71842)$
$\approx 1 - P(Z < 0.72)$
$= 1 - 0.76424$
$= 0.23576$

2. $N \sim$ Neg. Binomial (K, P)

$P(N = n) = {}^{n+2}C_n (0.9)^3 \, 0.1^n$
$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$

So, $K = 3$; $p = 0.9$

$M_y t = M_N(\log M t)$

$X \sim$ Gamma $(6, 2)$

$Y \to$ total claim amount

i) MGF of Y

$M_y(t) = E(E \cdot e^{yt} | N = n)$

$E(e^{yt} | N = n) = E(e^{x_1 t + x_2 t + x_3 t + \dots x_n t})$
$= E e^{t x_1} \cdot E e^{t x_2} \dots E e^{t x_n}$
$= \left(1 - \frac{t}{2}\right)^{-6n}$

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So, $E(E(e^{yt}/N=n)) = E\left((1-\frac{t}{2})^{-6n}\right)$

$= E(e^{N\log(1-t/2)^{-6}}$

$= M_N(\log(1-\frac{t}{2})^{-6})$

$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^K \Rightarrow \left(\frac{pe^{\log(1-t/2)^{-6}}}{1-qe^{\log(1-t/2)^{-6}}}\right)^K$

$= \left(\frac{p(1-t/2)^{-6}}{1-q(1-t/2)^{-6}}\right)^3$

$= \left(\frac{0.9(1-t/2)^{-6}}{1-0.1(1-t/2)^{-6}}\right)^3$

$V(Y) = E(N)V(X) + (E(X))^2 V(N)$

$= \frac{3}{0.9} \times \frac{6}{4} + \left(\frac{6}{2}\right)^2 \cdot \frac{3(0.1)}{(0.9)^2}$

$= 2.887 \Rightarrow$ st. deviation

3. $f(x) = \lambda e^{-\lambda(x-\alpha)}$

MGF: $E(e^{tx}) \Rightarrow \int_\alpha^\infty e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$

$M_X t = \lambda \int_\alpha^\infty e^{tx} e^{-\lambda x} e^{\lambda x} dx$

$= e^{\lambda\alpha} \lambda \int_\alpha^\infty e^{-(\lambda-t)x} dx$

$= \frac{e^{\alpha\lambda}}{t-\lambda} \left[e^{-(\lambda-t)x}\right]_\alpha^\infty = \frac{e^{\lambda\alpha}\lambda}{t-\lambda}\left[-e^{-(\lambda-t)\alpha}\right]$

$= \frac{e^{\lambda\alpha}\lambda}{\lambda-t} e^{-\lambda\alpha} e^{t\alpha} = \frac{e^{t\alpha}\lambda}{\lambda-t}$

(ii) $M_x t = \frac{\lambda e^{t\alpha}}{\lambda - t} = \lambda e^{t\alpha}(\lambda - t)^{-1}$

$M_x' t = \lambda[e^{t\alpha}(\lambda - t)^{-2} + (\lambda - t)^{-1} e^{t\alpha}(\alpha)]$

$= \lambda[(\lambda - t)^{-1} e^{t\alpha}(\alpha) + e^{t\alpha}(\lambda - t)^{-2}]$

$= \lambda e^{t\alpha}(\lambda - t)^{-2}[\alpha(\lambda - t) + 1]$

$= \lambda(\lambda)^{-2}[\alpha(\lambda) + 1]$

$= \frac{1}{\lambda}[\alpha\lambda + 1]$

(iii) $M_x''(t) = \lambda[\alpha e^{t\alpha}(\lambda - t)^{-2} + \alpha^2(\lambda - t)^{-1} \cdot e^{t\alpha} + e^{t\alpha}(+2)(\lambda - t)^{-3} + (\lambda - t)^{-2} e^{t\alpha}]$

put $t = 0$

$= \lambda[\alpha\lambda^{-2} + \alpha^2(\lambda)^{-1} + 2\lambda^{(-3)} + \alpha\lambda^{-2}]$

$= \lambda\left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3}\right]$

$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$

$V(x) = E(x)^2 - [E(x)]^2$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\left(\frac{\lambda\alpha + 1}{\lambda}\right) \cdot \frac{\lambda\alpha + 1}{\lambda}\right]$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda}\right)$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \alpha^2 - \frac{1}{\lambda^2} - \frac{2\alpha}{\lambda}$

$= \frac{1}{\lambda^2}$

4. $G_V(\theta) = \left(\frac{p}{1-q\theta}\right)^K$

$G_V(\theta) = \Sigma \theta^v \times P(V=v)$

$P(V=4) \Rightarrow \frac{\Gamma K+4}{\Gamma 5 \, \Gamma K} p^K q^K$

$= \frac{\Gamma K+4}{4! \, \Gamma K} p^K (1-p)^K$

$= \frac{(K+3)!}{(K-1)! \, 4!} p^K (1-p)^K$

5. $f(x,y) = ax(x+y)$ $\quad 0 < x < 5, \; 10 < y < 20$

(i) $P(4 > \frac{1}{3}x + 10)$

$a\left(\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right) = 1$

$a\left(\frac{1250}{3} + 1875\right) = 1$

$a = \frac{3}{6875}$

$a \int_{10}^{20} \int_0^5 (x^2 + xy)\, dx\, dy = 1$

$\int_{10}^{20} \left[\frac{x^3}{3} + \frac{x^2}{2} y\right]_0^5 dy = \frac{1}{a}$

$\int_{10}^{20} \frac{125}{3} + \frac{25}{2} y \, dy = \frac{1}{a}$

$\left(\frac{125}{3} y + \frac{25}{4} y^2\right)_{10}^{20} = \frac{1}{a}$

$833.33 + 2500 - (416.667 + 625) = \frac{1}{a}$

$a = 0.00043636$

(iii) $E(Y|x)$

$$f_{y/x} = \frac{f(x,y)}{f(x)}$$

$$f(x) = \frac{3}{6875}\int_{10}^{20} x^2 + xy \, dy$$

$$= \frac{3}{6875}\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= \frac{3}{6875}\left[20x^2 + x(400) - 10x^2 - 50x\right]$$

$$= \frac{30x(x+15)}{6875}$$

$$f_{y/x} = \frac{3}{6875}\frac{x(x+y)}{30x} \times 6875(x+15)$$

$$\frac{3}{6875}\frac{x(x+y)}{10x}\frac{6875}{x+15}$$

$$f_{Y/x} = \frac{x+y}{10(x+15)}$$

$$\text{So, } E(Y/X) = \int_{10}^{20} y\frac{(x+y)}{10(x+15)} dy = \frac{1}{10(x+15)}\int_{10}^{20} xy + y^2 \, dy$$

$$= \frac{1}{10(x+15)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$$

$$= \frac{1}{10(x+15)}\left[200x + \frac{8000}{3} - 50x - \frac{1000}{3}\right]$$

$$= \frac{1}{10(x+15)}\left[150x + \frac{7000}{3}\right]$$

$$= \frac{1}{x+15}\left[15x + \frac{700}{3}\right]$$

(ii) $P(Y > \frac{1}{3}x + 10)$

$$f(y) = \frac{3}{6875}\int_0^5 x^2 + xy \, dx = \frac{3}{6875}\left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^5$$

$$= \frac{3}{6875}\left[\frac{125}{3} + \frac{25}{2}y\right]$$

So, $$\frac{3}{6875}\int_{\frac{1}{3}x+10}^{20}\left(\frac{125}{3} + \frac{25y}{2}\right)dy \Rightarrow \frac{3}{6875}\left[\frac{125y}{3} + \frac{25y^2}{4}\right]_{\frac{1}{3}x+10}^{20}$$

$$= \frac{3}{6875}\left\{\frac{125(20)}{3} + \frac{25(400)}{4} - \frac{125}{3}\left(\frac{1}{3}x + 10\right) - \frac{25}{4}\left(\frac{1}{3}x + 10\right)^2\right\}$$

$$= \frac{3}{6875}\left\{\frac{2500}{3} + 2500 - \frac{125}{9}x - \frac{1250}{3} - \frac{25}{4}\left[\frac{1}{9}x^2 + 100 + \frac{2}{3}x\right]\right\}$$

$$= \frac{3}{6875}\left[\frac{2500}{3} + 2500 - \frac{125x}{9} - \frac{1250}{3} - \frac{25x^2}{36} - 625 - \frac{25}{6}x\right]$$

$$= \frac{3}{6875}[1250 + 18]$$

6. $X \sim Poi(\lambda)$ $\qquad Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t - 1)}$ $\qquad M_Y(t) = e^{\mu(e^t - 1)}$

So, $Z = X - Y$

$$M_Z(t) = E(e^{tZ}) = E(e^{t(X-Y)}) = E(e^{tX} \cdot e^{-tY})$$

$$= E(e^{tX}) \times E(e^{-(tY)}) = \frac{E(e^{tX})}{E(e^{tY})}$$

$$= \frac{e^{\lambda(e^t - 1)}}{e^{\mu(e^t - 1)}} = e^{(\lambda - \mu)(e^t - 1)}$$

By uniqueness of MGF of $Z = e^{(\lambda-\mu)(e^t - 1)}$

$Z \sim Pois(\lambda - \mu)$

Now, let $Z = X + Y$

$$E(e^{tZ}) = e^{\lambda(e^t - 1)} \times e^{\mu(e^t - 1)}$$

$$= e^{(\lambda+\mu)(e^t - 1)}$$

By uniqueness of MGF, $Z \sim Pois(\lambda + \mu)$

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			X		
		1	2	3	4
Y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

(i) $V(X/Y=2) = E(X^2/Y=2) - [E(X/Y=2)]^2$

$E(X^2/Y=2) = \dfrac{\sum x^2 P(X=x, Y=2)}{P(Y=2)}$

when $x=1$ when $x=2$ when $x=3$ when $x=4$

$\dfrac{1^2 P(X=1, y=2)}{P(y=2)} + \dfrac{2^2 P(x=2, y=2)}{P(y=2)} + \dfrac{3^2 P(x=3, y=2)}{P(y=2)} + \dfrac{4^2 P(x=4, y=2)}{P(y=2)}$

$= \dfrac{1(0)}{0.6} + \dfrac{4(0.3)}{0.6} + \dfrac{9(0.1)}{0.6} + \dfrac{16(0.2)}{0.6}$

$= 8.8333$

$E(X/Y=2) = \dfrac{\sum x P(X=x, Y=2)}{P(Y=2)}$

$\dfrac{1(0)}{0.6} + \dfrac{2(0.3)}{0.6} + \dfrac{3(0.1)}{0.6} + \dfrac{4(0.2)}{0.6}$

$= \dfrac{0.6 + 0.3 + 0.8}{0.6}$

$= 2.8333$

$V(X/Y=2) = 0.807$

(ii) $f(u,v) = \frac{48}{67}(2uv - u^2)$ $\quad 0 < u < 1 \; ; \; \frac{u}{2} < v < 2$.

$E(U/V=v) = ?$

$\frac{f(u,v)}{f(v)} = f(U/V=v)$

$$f(v) = \frac{48}{67}\int_0^1 2uv - u^2 \, du$$

$$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right] = \frac{16}{67}[3v-1]$$

$$\frac{f(u,v)}{f(v)} = \frac{48(2uv^2 - u^2) \times 67}{67 \times 16(3v-1)}$$

$$= \frac{3}{3v-1}(2uv^2 - u^2)$$

Now, $E(U/V=v) = \frac{3}{3v-1}\int_0^1 u(2uv^2 - u^2) \, du$

$$= \frac{3}{3v-1}\left[\frac{2u^3 v^2}{3} - \frac{u^4}{4}\right]_0^1$$

$$= \frac{3}{3v-1}\left[\frac{2v^2}{3} - \frac{1}{4}\right]$$

$$= \frac{3}{3v-1}\left[\frac{8v^2 - 3}{12}\right]$$

$$= \frac{8v^2 - 3}{4(3v-1)}$$

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8. $x \sim \text{gamma}(3, 2)$ $\qquad E(x) = 3/2$, $V(x) = 3/4$

$E(Y | X = x) = 3x+1$ $\qquad \text{var}(Y | X = x) = 2x^2 + 5$

$E(Y) = E(E(Y|X=x))$

$= E(3x+1)$

$= 3E(x) + 1$

$= 3 \cdot \frac{3}{2} + 1$

$= \frac{11}{2}$

$V(Y) = E(V(X/Y)) + V(E(X|Y))$

$= E(2x^2+5) + V(3x+1)$

$= 2E(x^2) + 5 + 9V(x)$

$V(x) = E(x^2) - [E(x)]^2$

$3/4 + 9/4 = E(x^2)$

$E(x^2) = 3$

$V(Y) = 2(3) + 5 + 9(3/4)$

$= 11 + 27/4$

$= \frac{71}{4}$

9. $E(X) = 3$; $V(X) = 4$

$E(Y) = 4$; $V(Y) = 1$

correlation $= \text{cov}(x, y) = -0.3\sqrt{4}$

$= -0.6$

(i) $\text{cov}(X, X+Y) = \text{cov}(X, X) + \text{cov}(X, Y)$

$= V(X) + \text{cov}(X, Y)$

$= 4 + (-0.6)$

$= 3.4$

(ii) $Var(Z) = Cov(X+Y, X+Y)$

$= Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y)$

$= V(X) + 2Cov(X,Y) + V(Y)$

$= 4 + 2(-0.6) + 1$

$= 3.8$

10. $f(x,y) = Kx^{-\alpha}e^{-y/\beta}$ $\quad 1 < x < \infty;\ 1 < y < \infty$

$\alpha > 1;\ \beta > 0$

$$K\int_1^\infty\int_1^\infty x^{-\alpha}e^{-y/\beta}\,dx\,dy = 1$$

$$= K\int e^{-y/\beta}\left[\frac{x^{-(\alpha-1)}}{-\alpha+1}\right]_1^\infty = 1$$

$$= \frac{K}{1-\alpha}\int e^{-y/\beta}[-1] = \frac{K}{\alpha-1}\int e^{-y/\beta}\,dy = 1$$

$$= \frac{K}{\alpha-1}\left[\frac{e^{-y/\beta}}{-1/\beta}\right]_1^\infty = 1$$

$$= \frac{-K}{\alpha-1}\beta\left(e^{-1/\beta}\right) = 1$$

$$K = \frac{\alpha-1}{\beta \cdot e^{-1/\beta}}$$