Probability & Statistics.

① Let x_i be claim size on home insurance
Let y_a be claim size or car insurance.

$$x \sim N\ (800, 100^2)$$
$$y \sim N\ (1200, 300^2)$$

$$P\ (y_1 + y_2 + y_3 > x_1 + x_2 + x_3 + x_4 + 800)$$
$$P\ (y_1 + y_2 + y_3 - (x_1 + x_2 + x_3 + x_4) > 800)$$
$$(y_1 + y_2 + y_3) - (x_1 + x_2 + x_3 + x_4) \sim N\ (3 \times 1200 - 4 \times 800,$$
$$300^3 (3) +$$
$$(100)^2 + \sim N (400, 310000)$$

$$P\left(z_1 > \frac{800 - 400}{\sqrt{310,000}}\right)$$

$$P(z > 0.71842) = 1 - P(z < 0.71842)$$
$$= 1 - P\ (z < 0.72)$$
$$= 1 - 0.76424$$
$$= 0.23576.$$

② $N \sim$ Negative Binomial (k, p)

$$P(N = n) = {}^{n+2}C_n\ (0.9)^3\ (0.1)^n.$$

so, $k = 3$ and $p = 0.9$.

$$M_y t = M_n(\log M_x t)$$

$x \sim$ Gamma $(6, 2)$

$y \rightarrow$ Total claim amount

a.) Mgf of y

$$M_y(t) = E\ (\Sigma e^{yt} / N = n)$$
$$E(e^{yt} / N = n) = E\left(e^{x_1 t + x_2 t + \dots x_n t}\right)$$

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$$= E(e^{tx_1}) \cdot E(e^{tx_2}) \ldots\ldots E(e^{tx_n})$$

$$= \left(1 - \frac{t}{2}\right)^{-6n}$$

So, $E(E e^{ty}/N = n) = E\left[\left(1 - \frac{t}{2}\right)^{-6n}\right]$

$$= E\left[e^{n \log (1 - t/2)^{-6}}\right]$$

$$= M_N \left[\log (1 - t/2)^{-6}\right]$$

$$M_N(t) = \left(\frac{pe^t}{1 - qe^t}\right)^k = \left[\frac{pe^{\log(1-t/2)^{-6}}}{1 - qe^{\log(1-t/2)^{-6}}}\right]$$

$$= \left[\frac{(0.9)\left(1 - \frac{t}{2}\right)^{-6}}{6 - 0.1\left(1 - \frac{t}{2}\right)^{-6}}\right]$$

i) $V(y) = E(N) V(x + [E(x)]^2 . V(N)$

$$= \left[\frac{3}{0.9} \times \frac{6}{4}\right] + \left(\frac{6}{2}\right)^2 \times \frac{3(0.1)}{(0.9)^2}$$

Standard deviation = 2.887.

③ i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$

$MGF = E_x(e^{tx})$

$$= \int_\alpha^\infty e^{tx} \lambda e^{-\lambda(x-\alpha)} \, dx$$

$$M_x t = \lambda \int_\alpha^\infty e^{tx} e^{-\lambda x} e^{\lambda\alpha} \, dx.$$

$$= e^{\lambda x} \lambda \int_\alpha^\infty e^{-(\lambda-t)x} \, dx.$$

$$= \frac{e^{\alpha\lambda}}{t-\lambda} \left[e^{-(\lambda-t)x} \right]^\infty$$

$$= \frac{e^{\lambda x} \lambda}{t-\lambda} \left[-e^{-(\lambda-t)\infty} \right]$$

$$= \frac{e^{\lambda\alpha} \lambda}{\lambda-t} e^{-\lambda x} e^{tx}$$

$$= e^{tx} \frac{\lambda}{\lambda-1}$$

ii) $M_x(t) = \dfrac{\lambda e^{tx}}{\lambda - t}$

$M_x^2(t) = \lambda \left[e^{tx} (\lambda-t)^2 + (\lambda-t)^{-1} e^{t\alpha} (\alpha) \right]$

$= \lambda (\lambda-t)^{-1} e^{tx} (\alpha) + e^{tx} (\lambda-t)^{-2}$

$= \lambda e^{tx} (\lambda-t)^{-2} (\alpha(\lambda-t)+1)$

$t = 0 \quad = \lambda\,(\lambda)^{-2}\,\alpha\,(\lambda)^{-1}$

$$= \frac{1}{\lambda}\,(\lambda\alpha + 1)$$

$$= \frac{\lambda\alpha + 1}{\lambda}$$

$$M_x''(t) = \lambda\,(\alpha e^{t\alpha}(\lambda - t)^{-2} + \alpha^2(\lambda - t)^{-1}e^{t\alpha} + e^{t\alpha}(t^2)(\lambda - t)^{-3}$$

But $t = 0$

$$\lambda\,[\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + (2)(\lambda)^{-3} + \alpha(\lambda^{-2})]$$

$$\lambda\left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3}\right]$$

$$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$$

$\cancel{\lambda(E}$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda} - \left[\alpha - \frac{1}{\lambda}\right]^2$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda} + \frac{2\alpha}{\lambda}\right)$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \alpha^2 - \frac{1}{\lambda^2} - \frac{2\alpha}{\lambda}$$

$$= \frac{1}{\lambda^2}$$

4) $g_V(t) = \left(\frac{p}{1-qt}\right)^k$

$g_V(t) = \sum t^v P(V=v)$

$P(V=4) = \frac{\sqrt{k+4}}{\sqrt{5}\,\sqrt{k}} p^k q^k$

$= \frac{\sqrt{k+4}}{4!\sqrt{k}} p^k q^4$

$= \frac{(k+3)!}{k-1} p^k (1-p)^4$

5) $f(x,y) = ax(x+y)$

i) $P(y > \frac{1}{3}x + 10)$

$$a \int_{10}^{n} \int_{0}^{5} (x^2 + xy)\,dx\,dy = 1$$

$$\int_{10}^{20} \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^5 dy = \frac{1}{a}$$

$$\int_{10}^{20} \left(\frac{125}{3} + \frac{25}{2}y\right) dy = \frac{1}{a}$$

$$\left[\frac{125}{3}y + \frac{25}{4}y^2\right]_{10}^{20} = \frac{1}{a}$$

$8.33.33 + 2500 = (416.664 + 625) \times \frac{1}{6}$

$a = 0.00043636$

iii) $P(y > \frac{1}{3},\ X = 10)$

$f(y) = \frac{3}{6875} \int_0^5 x^2 + xy \, dx$

$= \frac{3}{6875} \left[\frac{x^3}{3} + \frac{x^2 y}{2} \right]_0^5$

$= \frac{3}{6875} \left[\frac{125}{3} + \frac{25y}{2} \right]$

so, $\frac{3}{6875} \int_{1/3 \times 10}^{10} \left(\frac{125}{3} + \frac{25y}{2} \right) dy$

$= \frac{3}{6875} \left[\frac{125}{3}(10) + \frac{25}{4}(400) - \frac{125}{3}\left(\frac{1}{3} \times 10\right) - \frac{25}{4} \left(\frac{1}{3} \times 10\right)^2 \right]$

$= \frac{3}{6875} \left[\frac{125}{3}(10) + \frac{25}{4}(400) - \frac{125}{3}\left(\frac{1}{3} \times 10\right) - \frac{25}{4} \left(\frac{1}{3} \times 10\right)^2 \right]$

$= \frac{3}{6875} \left(\frac{1250}{3} + 2500 - \frac{125x}{9} - \frac{1250}{3} - \frac{25x^2}{36} - 65 - \frac{25x}{4} \right.$

iii) $E(y|x)$

$f(x)/x = \frac{f(x,y)}{f(x)}$

$f(x) = \frac{3}{6875}\int_{10}^{20} x^2 + xy \, dy$

$= \frac{3}{6875}\left[x^2y + \frac{xy^2}{2}\right)_{10}^{20}$

$= \frac{3}{6875}\left[20x^2 + x(200) - 10x^2 - 50x\right)$

$= \frac{3}{6875}\left[10x^2 + 150x\right]$

$= \frac{30x}{6875}(x+15)$

$f_y/y = \frac{3}{6875} x(x+y) \times \frac{6875}{30x}(x+15)$

so, $E(y/x) = \int_0^{\infty} y \frac{(x-y)}{10(x+15)} dy$

$= \frac{1}{10(x+15)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$

$= \frac{1}{10(x+15)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$

$$= \frac{1}{10(x+15)}\left[200x + \frac{8000}{3} - 50x - \frac{1000}{3}\right]$$

$$= \frac{1}{10(x+15)}\left[150x + \frac{7000}{3}\right]$$

$$= \frac{1}{x+15}\left[15x + \frac{700}{3}\right]$$

6) $x \sim Poi(\lambda)$ $\qquad y \sim Poi(\lambda)$

$M_x(t) = e^{\lambda(e^t-1)}$ $\qquad M_y(t) = e^{\lambda(e^t-1)}$

so $Z = x - y$.

$M_z(t) = E(e^{tz}) = E\left(e^{t(x-y)}\right)$

$= E\left(e^{tx} \cdot e^{-ty}\right)$

$= E(e^{tx}) \times E(e^{-ty})$

$= \frac{E(e^{tx})}{E(e^{ty})}$

$= \frac{e^{\lambda(e^t-1)}}{e^{y(e^t-1)}}$

$= e^{(\lambda-\lambda)(e^t-1)}$

MGF of $z = e^{(\lambda-\lambda)(e^t-1)}$ $\sim Pois(\lambda)$

let $z = x+y$

$E(e^{tz}) = e^{\lambda(e^t-1)} \times e^{y(e^t-1)}$

$= e^{(x+y)(e^t-1)}$

$z \sim Pois(\lambda+y)$

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		X			
		1	2	3	4
y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

? $V(X/y=2) = E(x^2/y=2) - [E(x/y=2)]^2$

$E(x^2/y=2] = \sum x^2 \frac{P(x=x, y=2)}{P(y=2)}$

where $x=1$; $x=2$, $x=3$, $x=4$

$(1)^2 \frac{P(x=1\ y=2)}{P(y=2} + (2)^2 \frac{(P(x=2, y=2)}{P(y=2} + (3)^2 \frac{P(x=3, y=2)}{P(y=2}$

$+ (4)^2 \frac{P(x=4\ y=2)}{P(y=2)}$

$\frac{(1)(0)}{0.6} + \frac{4(0.3)}{0.6} + \frac{9(0.1)}{0.6} + \frac{(16)0.2}{0.6}$

$E(x/y=2) = \sum_x x \frac{P(x=x, y=2)}{P(y=2}$

$\frac{1(0)}{0.6} + \frac{2(0.3)}{0.6} + \frac{3(0.1)}{0.6} + \frac{4(0.2)}{0.6}$

$= \frac{0.6+6+3+0.8}{0.6}$

$= 2.8333$

$\therefore v(x/y=2) = 8.7333 - (2.8333)^2$

$= 0.807$

ii) $f(u,v) = \frac{48}{67}(24v - v^2)$

$\frac{f(u,v)}{f(v)} = f(v/v - v$

$f(v) = \frac{48}{67}\int_0^1 (24v - v^2)\,du$

$= \cancel{68}\ \frac{48}{67}\left[v^2 v - \frac{u^3}{3}\right]_0^1$

$= \frac{48}{67}\left[v - \frac{1}{3}\right]$

$= \frac{16}{67}(3v - 1]$

$f(v) = \frac{16}{37}[3v - 1]$

$\frac{f(u.v)}{f(v} = \cancel{8}\ \frac{48}{67}\left(\frac{2uv^2 - v^2}{16(3v-1}\right) \times 67.$

$= \frac{3}{3v-1}(24v^2 - 4^2)$

$E(v/v=v) = \frac{3}{3v-1}\int_0^1 v(24v^2 - v^2)\,du$

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$$= \frac{3}{3v-1}\left[\frac{2v^3 v^2}{3} - \frac{u^4}{4}\right]_0^1$$

$$= \frac{3}{3v-1}\left[\frac{3v^2}{3} - \frac{1}{4}\right]$$

$$= \frac{3}{3v-1}\left[\frac{8u^2-3}{12}\right] = \frac{8v^2-3}{4(3v-1)}$$

2) $x \sim$ Gramma $(3, 2)$

$E(y|x = x) = 3x+1$

$E(y) = E(E\, y/x = x)]$

$= E(3x+1)$

$= 3E(x) + 1$

$= 3\left(\frac{3}{2}\right)+1$

$= \frac{9}{2}+1$

$= \frac{9+2}{2}$

$= \frac{11}{2}$

$v(y) = E(v(x/u)] + (v(E(x))u)$

$= E(2x^2+5) + v(3x+1)$

$= 2E(x^2) + 5 + 9v(x)$

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$v(x) = \mathcal{E}(x^2) - (\mathcal{E}(x))^2$

$\frac{3}{2} + \frac{9}{4} = (\mathcal{E}(x))^2$

$\mathcal{E}(x)^2 = 3$

$v(y) = 2(3) + 5 + 9\left(\frac{3}{4}\right)$

$= 11 + \frac{27}{4}$

$= \frac{44 + 27}{4}$

$= \frac{71}{4}$

9.) $\mathcal{E}x = 3$ $\quad v(x) = 4$

$\mathcal{E}y = 4$ $\quad v(y) = 1$

covelction coeff $= 0.3\sqrt{4} = \text{cov}(x,y)$

$= 0.6$

$Z = x + y$

i) $\text{Cov}(x, x+y) = \text{cov}(x, x) + \text{cov}(x,y)$

$= v(x) + \text{cov}(x,y)$

$= 4 + (-0.6)$

$= 3.4$

ii) $\text{Var}(z) = \text{cov}(x+y, x+y)$

$= \text{cov}(x,x) + 2\text{cov}(x,y) + \text{cov}(y,y)$

$= \text{cov}(x,x) + 2\text{cov}(x,y) + v(y)$

$= v(x) + 2\text{cov}(x,y) + 1$

$= 4 + 2(-0.6) + 1$

$= 5 - 1.2$

$= 3.8$

10.) $f(x,y) = kx^{-\alpha} e^{-y/\beta}$

$$k \int_1^\infty \int_1^\infty x^{-\alpha} e^{-y/\beta} \, dx \, dy = 1$$

$$k \int e^{-y/\beta} \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty = 1$$

$$\frac{k}{1-\alpha} \int e^{-y/\beta} [-1]$$

$$\frac{k}{\alpha-1} \int e^{-y/\beta} \, dy = 1$$

$$\frac{k}{\alpha-1} \left[\frac{e^{-y/\beta}}{-1/\beta} \right]_1^\infty - \frac{k}{\alpha-1} (\beta (e^{-1/\beta}) = 1$$

$$k = \frac{\alpha-1}{\beta(e^{-1/\beta})}$$