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ASSIGNMENT 1

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Q.1

(i) ~~Arbi~~ Arbitrage opportunity is when investors gain from price mismatch of the same underlying asset in two different markets. Arbitrage opportunity occurs as a result of market inefficiencies.

(ii) Law of one price states that any two portfolios which have same weights should also have the same price. If this doesnt hold then the investor can buy the underlying at cheap prices and sell it at a higher price to earn riskless profit.

(iii)

(a) $K = 120$ (put), $S_0 = 125$, $r = 0.05$, $t = 3$ months

Call = 30.

$q = 15\%$ p.a.

Bye put call parity,

$$p = c_t + Ke^{-r(T-t)} - S_t e^{-q(T-t)}$$

$$= 30 + 120e^{-0.05 \times 0.25} - 125e^{-0.15 \times 0.25}$$

$$p = 28.11.$$

(b) $p = 23$.

If the price of put option is 23, the investor can buy put at a cheaper price and sell call option at a higher price.

Since put call parity doesnot hold true the investor will make arbitrage profit.

Q.4]

Dividends are denoted by $d_1, d_2, d_3, \ldots d_n$.

If the option price is exercised prior to the ex-dividend date $= S_t - k$

If option is not exercised the price will drop to $S_t - d$.

The value of american option is greater than $S_t - d - ke^{-r(T-t)}$

It is never optimal to exercise the option if

$$S_t - d_n - ke^{-r(T-t)} \geq S_{(n)} - k.$$

using this,

$$k(1 - e^{-r(T-t)}) = 350\,(1 - e^{-0.95(0.83-0.25)})$$
$$= 18.87.$$

and.

$$65\,(1 - e - 0.95(0.83 - 0.25)) = 10.91.$$

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Q.2 $dX_t = \alpha u(T-t)dt + \sigma\sqrt{(T-t)}\,dZ_t$

Let,

$A_t = \alpha u(T-t)$, $B_t = \sigma\sqrt{(T-t)}$ — I .

$$dF = \frac{df}{dx} B_t\,dZ_t + \left(\frac{df}{dt} + \frac{df}{dx}A_t + \frac{1}{2}\frac{d^2f}{dx^2}B_t\right)dt$$

$$= -f B_t\,dZ_t + f\frac{dm}{dt}(T-t)\,dt - fA_t\,dt + \frac{1}{2}B_t^2\,dt$$

$$dF = f\left(\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}B_t^2\right)dt - fB_t\,dZ_t.$$

For f to be a martingale,

$$\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}B_t^2 = 0$$

$$\therefore \frac{dm}{dt}(T-t) = A_t + \frac{1}{2}B_t^2$$

$$\therefore \frac{dm}{dt}(T-t) = \alpha u - \frac{1}{2}\sigma^2 \quad (\text{from I})$$

[Q.3]

(i) $F(t,x) = e^{-t}x^2$

X is SDE

$$\frac{dX_t}{X_t} = 0.25\,dt + \sigma\,dW_t$$

$dY_t = ?$

$$\frac{df}{dt} = -e^{-t}x^2 = -f$$

$$\frac{df}{dx} = e^{-t}2x$$

By Ito's lemma,

$$dY_t = \frac{df}{dt}\cdot dt + \frac{df}{dx}\cdot dX_t + \frac{1}{2}\frac{d^2f}{dX^2}\cdot\sigma^2X_t^2\cdot dt$$

$$= -f\cdot dt + 2e^{-t}X_t\,dX_t + e^{-t}\sigma^2X_t^2\cdot dt$$

$$= -Y_t\cdot dt + 2e^{-t}X_t^2\cdot\frac{dX_t}{X_t} + e^{-t}\sigma^2X_t^2\,dt$$

$$= -Y_t\cdot dt + 2Y_t[0.25dt + \sigma dW_t] + \sigma^2Y_t\,dt$$

$$\therefore \frac{dY_t}{Y_t} = \left[2\times\frac{1}{4} + \sigma^2\right]dt + 2\sigma dW_t$$

$$= [\sigma^2 - 0.5]\,dt + 2\sigma dW_t.$$

$$\therefore dY_t = [\sigma^2 - 0.5]Y_t\,dt + 2\sigma Y_t\,dW_t.$$

(ii) The process is a martingale if drift is 0.
This means $\sigma^2 - 0.5 = 0$, $\sigma^2 = 0.05$

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Q.5

Let S be the stock price

$P(S > 258)$.

$\because S$ is a GBM

$$S_t = S_0\, e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

$$\therefore \ln(S_t) \sim N\left(\left(\ln(S_0) + (\mu - \sigma^2/2)t + \sigma W_t\right), \sigma^2 t\right)$$

$$\therefore \ln(S_t) \sim \phi\left(\ln 254 + (0.16 - 0.35^2/2)0.5,\ 0.35 \times 0.5^{1/2}\right)$$

$$= \phi(5.59, 0.0247)$$

$$\therefore P(S > 258) = 1 - N(5.55 - 5.59)/0.0247$$

$$= 1 - N(-0.1364)$$

$$= 0.5542.$$

The put option is exercised if the stock price less than rs. 258 in 6 months time.

The probability $= 1 - 0.5542$

$= 0.4457$.

Q.6

$$\ln\left(\frac{S_t}{S_0}\right) = \mu t + \sigma B_t$$

$$\frac{S_t}{S_0} = e^{\mu t + \sigma B_t}$$

$$Y = S_t = S_0 e^{\mu t + \sigma B_t}$$

(i) $$\frac{dS_t}{S_t} = x\,dB_t + y\,dt$$

By ito's lemma,

$$dS_t = \frac{dy}{dt}\cdot dt + \frac{dy}{dB_t}\cdot dB_t + \frac{d^2y}{dB_t}\cdot\frac{dB_t^2}{2}$$

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t + \frac{\sigma^2 S_t}{2}\,dt$$

$$= S_t\cdot dt\left(\mu + \frac{\sigma^2}{2}\right) + \sigma S_t \cdot dB_t$$

$$\frac{dS_t}{S_t} = x\,dB_t + y\,dt$$

$$x = \sigma, \quad y = \mu + \frac{1}{2}\sigma^2$$

(ii) $$E(S_T) = E\left(S_0 e^{\mu t + \sigma B_t}\right)$$

$$= S_0 e^{\mu t}\cdot E\left(e^{\sigma B_t}\right)$$

$B_t \sim N(0,1)$

$E(e^{\sigma B_t})$ is the MGF of Normal

$= e^{\mu + \frac{1}{2}\sigma^2 t^2}$

$= S_0 e^{\mu t} \left(e^{1/2 t \sigma^2} \right)$

$= S_0 e^{t(\mu + 1/2 \sigma^2)}$

$Var(S_T) = E(S_T^2) - [E(S_T)]^2$

$= E\left(S_0^2 \cdot e^{2\mu t + 2\sigma B t}\right) - \left[S_0 e^{t(\mu + 1/2\sigma^2)}\right]^2$

$= S_0^2 \cdot e^{2\mu t} \cdot E\left[e^{2\sigma B t} - \left(S_0 e^{t(\mu + 1/2\sigma^2)}\right)^2\right]$

$= S_0^2 e^{2\mu t} \left(e^{2\sigma^2} - e^{\sigma^2 t}\right)$

iii)

$Cov(S_{t_1}, S_{t_2}) = E(S_{t_1}, S_{t_2}) - E(S_{t_1})(E(S_{t_2}))$

$E(S_{t_1} S_{t_2}) = S_0 e^{\mu t_1} E(e^{\sigma B t_1}) \cdot S_0 e^{\mu t_2} E(e^{\sigma B t_2})$

$S_0^2 e^{\mu(t_1 + t_2)} \cdot E\left(e^{\sigma B t_1 + \sigma [B t_1 + (B t_2 - B t_1)]}\right)$

The Expectation sum is the MGF of Normal distribution.

Therefore,

$= S_0^2 e^{\mu(t_1 + t_2)} \cdot e^{2\sigma^2 t_1} \cdot e^{1/2 \sigma^2 (t_2 - t_1)}$

$= S_0^2 e^{\mu(t_1 + t_2)} \cdot e^{3/2 \sigma^2 t_1 + 1/2 \sigma^2 t_2}$

Q.9

(i) Forward price $= Se^{(rt)}$

S = stock price, t = delivery time, r = r.f rate.

put call parity: $C + Ke^{-rT} = P + S$

C and P = price of European call and put

K = Strike price, S = Stock price, t = time to expiry

$\therefore \quad 13.334 + 70e^{(-0.25r)} = 0.120 + S$
$\quad 8.869 + 75e^{(-0.25r)} = 0.568 + S$

By solving the above eqn.

$S = 82, \quad r = 0.07$

$\therefore F = 82\, e^{(0.07 \times 0.25)}$

$\therefore F = 83.45$

(ii)

$e^{(r_6 \times 0.5)} = e^{(r_3 \cdot 0.25)} \cdot e^{(r_f \cdot 0.25)}$

$r_3 = 0.07$

By substituting,
$2.569 + 90e^{(-0.5 \times r_6)} = 7.909 + 82$
$r_6 = 0.05$ and $r_f = 0.05$

(iii) By using put call parity,

$$6.899 + ae^{(-0.007 \times 0.25)} = 1.055 + 82$$
$$b + 80e^{(-0.007 \times 0.25)} = 1.789 + 82$$
$$2.594 + 85e^{(-0.007 \times 0.25)} = c + 82$$

By solving the equations we get.

$a = 77.5$
$b = 5.177$
$c = 4.119$.

Q10]

$\because$ Rate of interest is 0, the risk neutral probability will be

$$q = (1-d) / (u-d)$$

For a recombining tree,

$$d = \frac{1}{u}$$

$$q = \left(1 - \frac{1}{u}\right) \Big/ \left(u - \frac{1}{u}\right)$$

$$= \frac{1}{u+1}$$

For no arbitrage opportunity we must have $u > 1 > d$.

Then $u > 1 \Rightarrow u+1 > 2$

$$q = \frac{1}{u+1} < \frac{1}{2}$$

Hence proved.

(ii) from i, $q = \frac{1}{u+1}$

$q = 1/2$

$\therefore u = 2$

$\because u = e^{(\sigma\sqrt{\Delta t})}$

$\therefore \sigma = 240.11\%$.

[Q.8]

Current stock price $= S_0$
option price $= f$
Duration $= T$

In our portfolio we will have short position in the option and long position is Δ shares.

If there is a upward movement the value of portfolio becomes $\Delta S_0 u - f_u$

If there is a downward movement the value of the portfolio becomes $\Delta S_0 d - f_d$

The two portfolios will be equal if,

$$\Delta S_0 u - f_u = \Delta S_0 d - f_d$$

or $\Delta = \frac{f_u - f_d}{S_0 - S_0 d}$

so that the portfolio is risk free ~~as~~ therefore it must earn a risk free rate of interest.

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$\therefore$ p.v of portfolio = $(\Delta S_0 u - f_u) e^{-rT}$

The cost of building the portfolio is $\Delta S_0 - f$

If the portfolio grows at rf rate,

$$\Delta S_0 - f = (\Delta S_0 u - f_u) e^{-rt}$$

$$\therefore f = \Delta S_0 - (\Delta S_0 u - f_u) e^{-rT}$$

$$\therefore f = e^{-rT} [p f_u + (1-p) f_d]$$

where $p = [e^{rT} - d] / [u - d]$

(ii) ~~The~~ If there is an upward movement in the stock price the value of call option will increase and the value of put option decreases when the ~~pr~~ stock price goes down. This happens because ~~the value~~ ~~of~~ we are calculating the value of option in terms of the value of underlying asset.

(iii) $E(S_T)$ at time T

$$= p S_0 u + (1-p) S_0 d \cdot 0.5$$

or.

$$E(S_T) = p \cdot S_0 (u - d) + S_0 d - 0.5$$

Substituting p.

$$E(S_T) = e^{rT} \cdot S_0$$

By this we can interpret that the stock price grows at the risk free rate.

(iv) –

[Q.11]

(i) A recombining binomial Tree is one in which the sizes of ups and down steps is the same under all steps and across all intervals. therefore the risk neutral probability 'q' is also constant at all times and in all states.
It has only (n+1) possible states of time as opposed to second possible states in a non recombining.
It assumes that the volatility and drift parameters of an underlying are constant over time.

(b1) (a)

1000 → 1100 → 1320, 990

1000 → 950 → 1140, 855

$$q_1 = \frac{e^{(0.0175)} - 0.95}{1.10 - 0.95}$$

$= 0.4570$

$$q_2 = \frac{e^{0.025} - 0.90}{1.20 - 0.90}$$

$= 0.41772$

put payoffs at 4 states are 0, 0, 0, 95.
The value of option $V_1(1)$ following an up step over the first 3 month is

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$V_1 \cdot e^{0.0025} = [0.41772 \times 0] + [0.58228 \times 0]$

$\therefore V_1(1) = 0$

The value of option following a down step over the first three months is

$V_2(2)\ e^{0.0025} = [0.41772 \times 0] + [0.58228 \times 95]$

$= 53.9502$

The current value is

$V_0\ e^{(0.00175)} = [0.4510 \times 0] + [0.5490 \times 53.9508]$

$= 29.105$

Q. 12

(i) $Z(t)$ is a standard Brownian.

(a) $du(t) = 2dZ_t - 0$

$= 0dt + 2dZ_t$

Thus $u(t)$ has zero drift.

(b) $dV(t) = d[Z(t)]^2 - dt$

$d[Z(t)]^2 = 2Z(t) \cdot dZ_t + dt$

Thus, $dV(t) = 2Z_t \cdot dZ_t$. Thus stochastic process $V(t)$ has 0 drift.

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(c) $dW(t) = d[t^2 Z(t)] - 2t Z(t)\, dt$

~~Beca~~ $\because d[t^2 Z(t)] = t^2 dZ(t) + 2t Z(t)\, dt$

$dW(t) = t^2 dZ(t)$

$\therefore$ The process $W(t)$ has 0 drift

[Q013]

Let $\frac{S_t}{S_0} \sim LN\left[\left(\mu - \frac{1}{2}\sigma^2\right)\cdot t,\ \sigma^2 t\right]$

expected return $= \mu$
volatility $= \sigma$

Expected value at first time step on the tree $= q S_0 u + (1-q) S_0 d$

$S_0 e^{\mu dt} = q S_0 u + (1-q) S_0 d$

$q = (e^{\mu dt} - d) / (u - d)$

Standard deviation of stock price $= \sigma\sqrt{\delta t}$

The variance is,

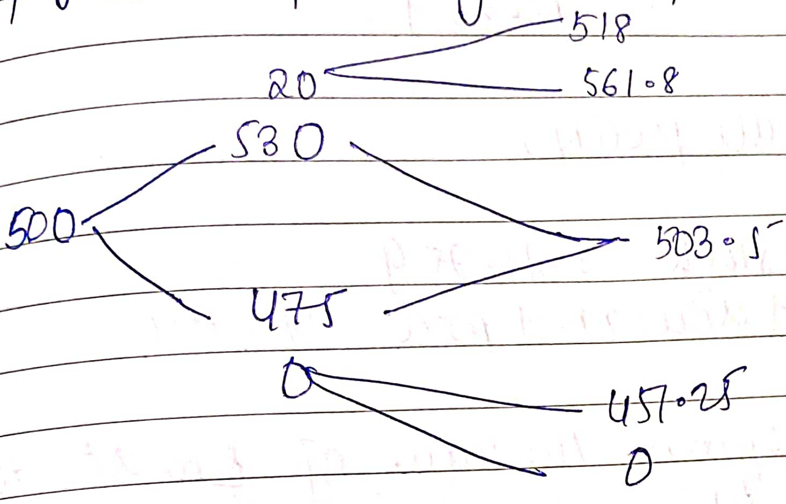
$q u^2 + (1-q) d^2 - [q u + (1-q) d]^2 = \sigma^2 dt$

$\therefore e^{\mu dt}(u+d) - ud - e^{2\mu dt} = \sigma^2 dt$

$u = e^{\sigma\sqrt{dt}}$

[2014]

(i) payoffs Diagram for European call.



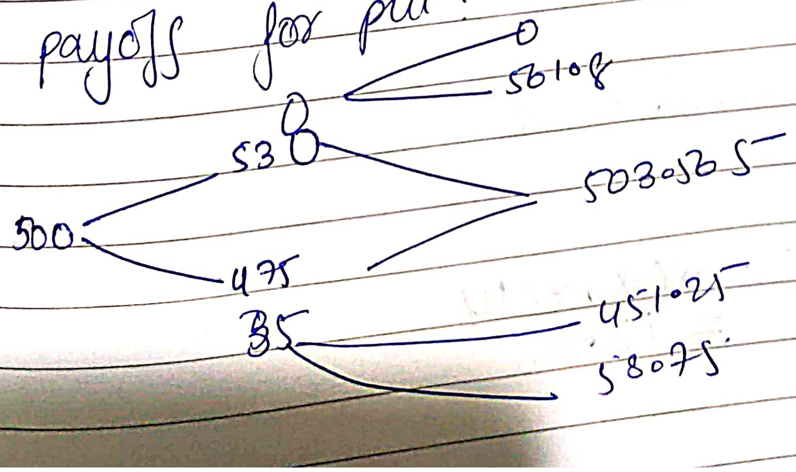
$u = 1.06$, $d = 0.95$, $r = 0.005$, $t = 0.25$

$p = e^{(rt)} - d \,/\, (u - d)$
$= 0.05689$

$1 - p = 0.4311$

Value of European call $= e^{(-0.5 \times 0.005)} \times 518 \times 0.5689^2$
$= 16.351$

(ii) payoffs for put.



Value of European put $= e^{(-0.05 \times 0.5)} \times 0.4311^2 \times 58.75$
$\times 2 \times 0.431 \times 0.5689 \times 6.50$

$= 13.759.$

Using put call parity,

Value of put + stock $= 513.759$
Value of calls + discounted price $= 513.759.$

(iii) For American option, the value of ~~fo~~ at the final node is same for a European option.

Value at Node B $= 6.50 \times e^{(-0.25 \times 0.05)} \times 0.4311$
$= 2.767.$

Value at Node C $= e^{(-0.25 \times 0.05)} (6.50 \times 0.5689$
$+ 58.75 \times 0.4311)$
$= 28.66442$

Value of American put option at time $t=0$

$= e^{(-0.05 \times 0.25)} (35 \times 0.4311 + 2.767 + 0.5869)$
$= 16.46.$

(IV) real rate of return $= 0.09$
$p = 0.6614.$

Expected payoff $= 20 \times 0.6614$
$= 13.228$

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Q.15

(i) $f = f(S_t, t) = S^k$.

$\frac{df}{dS} = kS^{k-1}$, $\frac{d^2f}{dS^2} = k(k-1)S^{k-2}$

$\frac{df}{dt} = 0$.

$df = kS^{k-1}\, dS_t + \frac{1}{2} k(k-1)S^{k-2}(dS_t)^2$

$= f\left[k\mu + \frac{1}{2}k(k-1)\sigma^2\right]dt + f\left[k\sigma\right]dW_t$

Hence,

$f = S^k$ is GBM

with drift $= k\mu + \frac{1}{2}k(k-1)\sigma^2$

volatility $= k\sigma$

$f_t = f_0 e^{(\mu - 1/2\sigma'^2)t + \sigma' W_t}$

This means $\frac{f}{f_0}$ is $LN\left[\left(\mu', \frac{1}{2}\sigma'^2\right)t, \sigma'^2 t\right]$

$E(f) = f_0 e^{\mu_1 t}$, $V(f) = f_0\, e^{2\mu' t}\left(e^{\sigma_1^2 t} - 1\right)$

(ii) Let $f = f(t, S_t) = e^{-rt} S_t$

$\frac{df}{dt} = -re^{-rt}S_t$, $\frac{df}{ds} = e^{-rt}$, $\frac{d^2f}{ds^2} = 0$

By Ito's calculus.

$df = -re^{-rt} S_t \cdot dt + e^{-rt} dS_t$.

$= (\mu - r) f\, dt + \sigma f\, dW_t$

which is a martingale if $\mu = r$.

(iii) S^k will be a martingale if $k\mu + \frac{1}{2} k(k-1)\sigma^2 = r$