# Assignment - I
## Probability & Statistics - II

1. $x = 200$ $\quad x \sim poi(\theta)$

$$\frac{x-\theta}{\sqrt{\theta}} \sim N(0,1)$$

$P[x_1 < \theta < x_2] = 0.95$

$P[-1.96 < \theta < 1.96] = 0.95$

$\therefore$ 95% $CI = \theta \pm 1.96\sqrt{\theta}$

$= (172.281)$ & $(227.718)$.

2. $x_i = 1, 2, 3, \ldots, n$ ; iid RV

Mean $= \mu$ $\quad$ variance $= \sigma^2$.

$$\bar{x} = \frac{\Sigma x_i}{n} \Rightarrow S^2 = \frac{1}{n-1} \sum^{n} (x_i - \bar{x})^2.$$

i) for $\bar{x}$ mean

$$E(\bar{x}) = E\left(\frac{\Sigma x_i}{n}\right) = \frac{1}{n} E(\Sigma x_i)$$

$$= \mu.$$

$$v(\bar{x}) = v\left(\frac{\Sigma x_i}{n}\right) = \frac{\Sigma v(x_i)}{n^2}$$

$$= \frac{\sigma^2}{n}$$

ii) $$E(S^2) = E\left(\frac{1}{(n-1)} \Sigma (x_i - \bar{x})^2\right).$$

$$= \frac{1}{n-1} \Sigma \left[E(x_i^2) + E(\bar{x}^2) - 2\bar{x}^2\right].$$

$$= \frac{1}{n-1} \Sigma \left[ \sigma^2 + \mu^2 - \left( \frac{\sigma^2}{n} + \mu^2 \right) \right]$$

$$= \frac{1}{n-1} \Sigma \left\{ \sigma^2 \left( \frac{n-1}{n} \right) \right\}.$$

$$= \frac{\cancel{n}\sigma^2}{\cancel{n}}$$

$$= \sigma^2.$$

iii) . $X_i \sim$ Norm

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$V(\chi^2_{n-1}) = \frac{n-1}{2(1/n)} = 2(n-1).$$

$$\Rightarrow \frac{(n-1)^2}{\sigma} V(s^2) = 2(n-1)$$

$$\Rightarrow V(s^2) = \frac{2\sigma^4}{n-1}$$

ii) $n=10$ $\qquad \sigma^2 = 100$

$$P[50 < s^2 < 150] = P\{s^2 < 150\} - P\{s^2 < 50\}.$$

$$= P\{\chi^2_4 < 13.5\} - P[\chi^2_4 < 4.5].$$

$$= 0.7342.$$

3. $X$: no. of claims. $\qquad X \sim Bin(4, p)$.

| no. of claims | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| % policies | 86 | 75 | 16 | 2 | 1 |

i) $L = \prod_{i=1}^{180} P(X=x)$

$= [P(X=0)]^{86} \cdot [P(X=1)]^{75} \cdot [P(X=2)]^{16} \cdot [P(X=3)]^{2} \cdot [P(X=4)]^{1}$

$L = \text{constant} \times (1-p)^{603}$

$\frac{d}{dp} \ln L = \frac{-603}{1-p} + \frac{117}{p} = 0$.

$\Rightarrow 603p = 117(1-p)$.

$\Rightarrow p = 0.1625$.

$\frac{d^2}{dp^2} \ln L = \frac{-603}{(1-p)^2} - \frac{117}{p^2} < 0$

So minima.

ii) Adjusted Table.

| $O_i$ | $E_i$ | $\frac{(O_i - E_i)^2}{E_i}$ |
|---|---|---|
| 86 | 88.5547 | 0.0737 |
| 75 | 68.729 | 0.5722 |
| 19 | 22.7159 | 0.6078 |

$\chi^2_{cal} = \sum \frac{(O_i - E_i)^2}{E_i} = 1.2537$

$\chi_{tab} = \chi^2_1 = 3.841$

4. ii) Sample $= X_1, X_2, \ldots, X_n$.

Using method of moments.

$\Rightarrow E(X) = \frac{\Sigma X_i}{n}$

$\Rightarrow \frac{\beta}{c-1} = \bar{X}_n$.

$\Rightarrow \hat{\beta} = 2\bar{X}_n$

Unbiasedness verification.

$\Rightarrow E(g(x)) - \theta$

$E(2\bar{X}_n) - \beta$.

$2E\left(\frac{\Sigma X_{i}}{n}\right) - \beta$.

$\frac{2}{n}\Sigma\left(\frac{\beta}{2}\right) - \beta$

$\frac{2n\beta}{2n} - \beta$.

$\Rightarrow 0$.

it is an unbiased estimator.

i) $f(x) = \frac{c\beta^3}{(x+\beta)^4}$ $\quad x > 0 \quad \beta > 0$. —— (1)

From table book

$f(x) = \frac{\alpha\lambda^{\alpha}}{(\lambda + x)^{\alpha+1}}$ —— (2).

comparing ① & ② =)

$c = 3$

mean =) $\left[E(x) = \frac{B}{c-1} = \frac{B}{2}\right]$

Variance =) $\left\{v(x) = \frac{cB^2}{(c-1)^2(c-2)} = \frac{3B^2}{4}\right]$

iii) MSE = Var + Bias$^2$

$Var(\hat{B}) = v(b\bar{x}_n) = b^2 v(\bar{x}_n)$

$= \frac{3b^2B^2}{4n}$

$E(\hat{B}) = E(b\bar{x})$

$= \frac{b}{n}\left[n \frac{B}{2}\right]$

$= \frac{bB}{2}$

Bias $= \frac{bB}{2} - B = \frac{B(b-2)}{2}$

iv)

iv).

for coherence.

$$\text{Bias} = E(g(x)) - \theta$$
$$= E(\bar{B}) - B$$
$$= B\left[\frac{b-2}{2}\right]$$
$$= B\left[\frac{\frac{2n}{3+n} - 2}{2}\right] \quad \text{---} \left[b = \frac{2}{1+3/n}\right]$$
$$= B\left[\frac{2n-6-2n}{2(n+3)}\right]$$
$$= \frac{-3B}{n+3}$$

$$\therefore \text{MSE} = \frac{3B^2}{l+3}$$

$n \to \infty$, MSE $\to 0$ $\quad \therefore$ consistent.

5.

i). let $X_1, X_2, X_3 \ldots X_n$ follow $U(a,b)$
$$\bar{X} = \frac{\Sigma X_i}{n}$$

by MOM

$$E(X) \Rightarrow \frac{1}{2}(a+b) = \bar{x}$$

$$\Rightarrow (a+b) = 2\bar{x}$$

$$v(x) = s^2.$$

$$\frac{1}{12}(b-a)^2 = s^2.$$

$$\Rightarrow \frac{b-a}{2\sqrt{3}} = s.$$

$$\Rightarrow b-a = 2\sqrt{3}\,s.$$

$$\therefore (a+b)-(b-a) = 2\bar{x} - 2\sqrt{3}\,s.$$

$$2a = 2\bar{x} - 2\sqrt{3}\,s.$$

$$\underline{\underline{a = \bar{x} - \sqrt{3}\,s.}}$$

ii) $b = \bar{x} + \sqrt{3}\,s.$

iii). sample: 5, 12, 20, 35, 50

$\bar{x} = 24.4$ $\qquad$ $s = 18.147$.

$\hat{a} = \bar{x} - \sqrt{3}\,s.$ $\qquad$ $\hat{b} = \bar{x} + \sqrt{3}\,s.$

$\hat{a} = -7.078$ $\qquad$ $\hat{b} = 55.87.$

iv).

Sample : $x_1, x_2 \ldots x_n$. $\qquad$ $a = 0$.

$$L = \prod_{i=1}^{4} f(4).$$

$$= \left(\frac{1}{b-a}\right)^4$$

$$= \left(\frac{1}{b}\right)^4.$$

$$\ln L = -n \ln b$$

$$\frac{d}{db}(\ln L) = -\frac{n}{b}$$

$$\hat{b} = Max(x_i)$$

6.

7.

i)

Sample Mean:

Pri hosp $(\bar{x}) = \frac{\Sigma x_i}{n} = 35.056$

Pub hosp $(\bar{x}) = 30$

Sample Variance:

$$s^2_{Pr} = \frac{\Sigma x_i^2 - n\bar{x}}{n-1} = 67.403$$

$$s^2_{Pu} = 64.3125$$

ii)

$H_0 : \sigma_1^2 = \sigma_2^2$ $\qquad$ $H_1 : \sigma_1^2 \neq \sigma_2^2$

iii) ~~#~~

This is a one sided out.

$$t_{cal} = \frac{(\bar{X}_{pa} - \bar{X}_{pu}) - (\mu_{pa} - \mu_{pu})}{S_p \sqrt{1/n_1 + 1/n_2}}$$

$= 1.2363$

$t_{tab} = 1.746$

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i) An unbiased estimator is whose mean value is equal to the true parameter value ($\theta$).

ii) it is the measure of the difference between the expected value of the estimator & the true parameter value.

vi) a) Method of Moments.
b) MLE

9.

i). $t_K$ is defined as:

$$\frac{Z}{\sqrt{C/K}} \sim t_K \qquad \text{where } Z \sim N(0,1).$$

where K denotes the degrees of freedom.

ii) Mean = 0

Variance = $\frac{K}{K-2}$

$\therefore K > 2$

iii) $n = 10$ $\qquad$ $x = 50$.

$s^2 = 48.667$

10.

i) Type 1 error is probability of rejecting $H_0$; when it is actually true.

$H_0 = \mu = 3$ ; if $x < 3\cdot00$ $\quad H_0 \rightarrow$ ~~accepted~~ accept.

ii).

To verify:

$H_0 : \sigma_1^2 = \sigma_2^2$ ; $\qquad H_1 : \sigma_1^2 \neq \sigma_2^2$.

$$\frac{S_{Pr}^2 / S_{PV}^2}{\sigma_{Pr}^2 / \sigma_{PV}^2} \sim F_{n_{Pr}-1,\, n_{PV}-1}$$

for 95% CI.

$$\Rightarrow P\left[\frac{1}{x} < F_{8,18} < x\right] = 0.95$$

$$\Rightarrow P\left[\frac{1}{4.433} < \frac{67.403\,\sigma_{PV}^2}{64.3125\,\sigma_{Pr}^2} < 4.433\right]$$

iii)

$H_0 : \mu_{pr} = \mu_{pv}$ ; $H_1 : \mu_{pr} > \mu_{pv}$.

This is a one sided out.

$$t_{cal} = \frac{(\bar{X}_{pr} - \bar{X}_{pv}) - (\mu_{pr} - \mu_{pv})}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$= \frac{5.056}{8.6756\sqrt{2/9}} =$$

$$= \frac{5.056 \times 3}{8.6756\sqrt{2}}.$$

$$t_{cal} = 1.2363$$

$$t_{tab} = t_{16}, 5\%.$$

$$= 1.746.$$

∴ we have insufficient evidence to reject $H_0$ at 5% level of significance.

6.

ii). Type 2 error is the ~~probe~~ prob of accepting $H_0$ when it's supposed to be rejected.

i) - Type 1 error in the case is

$\Rightarrow P(X > 3100 / \lambda = 3000)$.

$\Rightarrow P\left(Z > \frac{3100 - 3000}{\sqrt{3000}}\right)$.

$\Rightarrow P(Z > 1.8257)$

$\Rightarrow 1 - P(Z < 1.8257)$.

$\Rightarrow 0.0356$.

iii)

Power of a test is the probability of rejecting $H_0$ when it's false.

Power of test $= P(X > 3100 / \bar{X} \approx N(n\lambda, n\lambda))$.

$= 1 - P(X < 3100 / \bar{X} \approx N(n\lambda, n\lambda))$.

$= 1 - P\left(Z < \frac{3100 - n\lambda}{\sqrt{n\lambda}}\right)$.

iv): $\bar{X} \sim N\left(\lambda, \frac{\lambda}{n}\right)$.

$\frac{\bar{X} - \lambda}{\sqrt{n/\lambda}} \sim N(0,1)$.

$\Rightarrow P\left[x_1 < \frac{\bar{x}-\mu}{\sqrt{\mu/\mu}} < x_2\right] = 0.99.$

$\Rightarrow P[-2.5758 < z < 2.5758] = 0.99$

$\Rightarrow 99\%\ CI = \frac{2900}{1000} = 2.5758\sqrt{\frac{2.9}{1000}}$

11.

$\Sigma x = 21320$

$n = 12$

$S = 10144$

$\bar{x} = 17767$

$\Sigma x^2 = 4919860000$

so, now,:

$\Sigma x_{new} = \Sigma x$ - incorrect val. + correct value

$= 21320 - (11000 + 48000) + (27500 + 31500).$

$= 213200.$

$\bar{x}_{new} = \frac{\Sigma x_{new}}{n}$

$= \frac{213200}{12}$

$= 17767.$

$$\Sigma x_{i_{new}}^2 = \Sigma x_i^2 - (11000^2 + 48000^2) + 27500^2 + 31500^2$$

$$= 4{,}243{,}368{,}010.$$

$$s^2 = \frac{1}{n-1}\left(\Sigma x_{i_{new}}^2 - n\bar{x}_{new}^2\right).$$

$$= \frac{1}{11}\left[4243360000 - (12 \times 17767^2)\right]$$

$$s^2 = 41396776.$$

$$s = \sqrt{41396776}.$$

$$s = 6434.$$