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Roll No

PROBABILITY & STATISTICS - 1

42.

Assignment - 1.

Q.1.] Calculate mean, median, mode, standard deviation and inter quartile range of the following data.

i) 4, 7, 9, 9, 11, 15, 18, 18, 18, 25

$n = 10$

$$\text{Mean} = \frac{\text{Sum of all the terms}}{\text{No. of terms}} = \frac{4+7+9+9+11+15+18+18+18+25}{10}$$

$$= \frac{134}{10}$$

$$\text{Mean} = 13.4.$$

$n = 10$ (even).

$$\text{Median} = \left[\frac{\frac{n}{2} + \frac{n+2}{2}}{2}\right]^{th} \text{term} = \frac{\frac{10}{2} + \frac{12}{2}}{2} = \frac{5+6}{2} = \frac{11}{2} = 5.5^{th} \text{ term}$$

$$= 5^{th} \text{ term} + 0.5\,(6^{th} \text{ term} - 5^{th} \text{ term})$$

$$= 11 + 0.5\,(15-11)$$

$$= 11 + 2$$

$$\text{Median} = 13$$

Mode is the most frequent occurring term.

18 occurs 3 times in the given data set.

$\therefore$ Mode = 18

$$\text{Standard deviation (s.d)} = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}}$$

$$\therefore S.D = \sqrt{\frac{(4-13.4)^2+(7-13.4)^2+(9-13.4)^2+(9-13.4)^2+(11-13.4)^2+(15-13.4)^2+(18-13.4)^2+(18-13.4)^2+(18-13.4)^2+(25-13.4)^2}{10}}$$

$$= \sqrt{\frac{(-9.4)^2+(-6.4)^2+(-4.4)^2+(-4.4)^2+(-2.4)^2+(1.6)^2+(4.6)^2+(4.6)^2+(4.6)^2+(11.6)^2}{10}}$$

$$= \sqrt{\frac{88.36+40.96+19.36+19.36+5.76+2.56+21.16+21.16+21.16+134.56}{10}}$$

$$= \sqrt{\frac{374.4}{10}}$$

$$= \sqrt{37.44}$$

$$= 6.118823$$

Inter-quartile Range

$Q_3 = \frac{3}{4}(n+1) = \frac{3 \times 11}{4} = 8.25^{th}$ term

$= 8^{th}$ term $+ 0.25(9^{th}$ term $- 8^{th}$ term$)$

$= 18 + 0.25(18-18)$

$= 18$

$Q_1 = \frac{1}{4}(n+1)^{th}$ term $= \left(\frac{1}{4} \times 11\right)^{th}$ term $= 2.75^{th}$ term

$= 2nd$ term $+ 0.75(3^{rd}$ term $-$ 2nd term$)$

$= 7 + 0.75(9-7)$

$= 7 + 0.75(2)$

$= 7 + 1.5$

$= 8.5$

$\therefore$ Interquartile Range $= Q_3 - Q_1 = 18 - 8.5$

$= 9.5$

Mean = 13.4 Standard deviation = 6.118823

Median = 13 Interquartile Range = 8.5.

Mode = 18

ii)

Number of households (x_i)	No. of people (f_i)	$x_i f_i$	Cumulative frequency	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
0	5	0	5	-2	4
1	11	11	16	-1	1
2	26	52	42	0	0
3	15	45	57	1	1
4	3	12	60	2	4
	60	120			10

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{120}{60} = 2$$

~~Size of~~

$$\text{Median} = \frac{\Sigma f_i}{2} = \frac{60}{2} = 30^{th} \text{ item}$$

$\therefore$ Median = $\cancel{1}$ 2 [which corresponds with the 30^{th} item]

Mode is the value of x_i with the highest frequency

$\therefore$ Mode = 2.

$$\therefore \text{Standard deviation} = \sqrt{\frac{(x_i - \bar{x})^2}{n}} = \sqrt{\frac{(x_i - \bar{x})^2}{\Sigma f_i}}$$

$$= \sqrt{\frac{10}{60}}$$

$$= 0.408248$$

Interquartile Range

$$Q_3 = \frac{3(n+1)}{4} = \frac{3 \times 61}{4} = 45.75^{th} \text{ item}$$

with corresponds with 3 households

$\therefore Q_3 = 3$

$$Q_1 = \frac{1(n+1)}{4} = \frac{1 \times 61}{4} = 15.25^{th} \text{ item}$$

which corresponds with 1 household.

$\therefore Q_1 = 1$

$\therefore$ Interquartile Range

$$Q_3 - Q_1 = 3 - 1 = 2$$

$\therefore$ Mean = 2

Median = 2

Mode = 2

Standard deviation = 0.408248

Interquartile Range = 2.

Q.2] It is assumed that claims arising on an industrial policy can be modelled as a Poisson process at a rate of 0.5 per year.

$\therefore$ X – r.v (No. of claims)

$\therefore X \sim P(0.5)$

i) $P(X=0) = ?$

$\therefore P(X=0) = P(X \leq 0)$

$= \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$

$= \frac{e^{-0.5} \cdot (0.5)^{0}}{0!}$

$= 0.60653$ – [from table book].

ii) Determine the Probability that, there is one or more claims in one of the years and no claims in each of the other two years.

Let 0 claims be a failure and one or more claims be a success.

Y – year with claims. X – no. of claims.

$P(X=0) = 0.60653$ – from (i)

$\therefore q$ (failure) $= 0.60653$

$p = 1 - q$

$= 1 - 0.60653$

$= 0.39347$.

$\therefore Y \sim Bin(3, 0.39347)$

$\therefore P(Y=1) = {}^{3}C_{1} \cdot p^{1} q^{2}$

$= \frac{3!}{2!\,1!} (0.39347)(0.60653)^{2}$

$= 3 (0.39347)(0.60653)^{2}$

$= 0.4342247$

iii) A claim has occurred.

To find:- Probability that next claim occurs after 2 years

T = time unit next claim

$\therefore T = 2$.

As we have to find the probability that the next claim will happen after 2 years.

$$T \sim Exp(0.5).$$

$$\therefore P(\cancel{x} > 2) = ?$$

$$\therefore P(\cancel{T} > 2) = 1 - P(X \leq 2)$$

$$= 1 - (1 - e^{-\lambda x})$$

$$= \cancel{1} - \cancel{1} + e^{-\lambda x}$$

$$= e^{-0.5 \times 2}$$

$$= e^{-1}$$

$$P(T > 2) = 0.3678794$$

Q.3] An analyst is interested in ussing gamma distribution with parameters $\alpha = 2$ and $\lambda = 1/2$.

The range of x is defined as $0 < x < \infty$

i.e $\boxed{X \sim Gamma(2, 1/2)}$

i) Mean = ? ~~Var~~ Standard deviation = ?

$$Mean = E(X) = \frac{\alpha}{\lambda} = \frac{2}{1/2} = 2 \times 2 = 4$$

$$Va(X) = \frac{\alpha}{\lambda^2} = \frac{2}{(1/2)^2} = 2 \times 4 = 8$$

$$\therefore \text{Standard deviation} = \sqrt{V(X)} = \sqrt{8} = \sqrt{4 \times 2}$$

$$= 2\sqrt{2}$$

ii) Comment briefly on it shape.

It will be a rightly skewed curve.

iii) Calculate the cumulative distribution function.

$\therefore \cancel{F(X \leq x)} \quad F_X(x) = P(X \leq x)$

It can be found out with the following relation.

$$2\lambda X = \chi^2_{\mu = 2\alpha}$$

$$\therefore F_X(x) = P(2\lambda X \leq 2\lambda x)$$

$$= P(\chi^2_{\mu=2\alpha} \leq 2\lambda x)$$

Given $\lambda = 1/2$, $\alpha = 2$.

$$\therefore F_X(x) = P(\chi^2_{\mu=4} \leq x) \quad //$$

Q.4.] In an actuarial office, there are 10 male and 8 female staff. 6 of the males and 7 of the females are fully qualified actuaries. Calculate the probability that a team of two workers randomly selected consists of two qualified females given that it contains at least one female.

M-male F-female. QF Qualified. female QM-Qualified Male

$\therefore P(M) = 10/18.$ $P(F) = 8/18.$

$P(QF) = 7/18$ $P(QM) = 6/18.$

$$P(\text{Atleast one female}) = {}^2C_1 (8/18)^1 (10/18)^1 + {}^2C_2 (8/18)^2 (10/18)^0$$

$$= \frac{2!}{1!\,1!}\left(\frac{8}{18}\right)\left(\frac{10}{18}\right) + \frac{2!}{0!\,2!}\left(\frac{8}{18}\right)^2 \cancel{\frac{10}{18}}$$

Q - Qualified female $Q \sim B(2, 7/18)$

$$P(\text{2 qualified female actuaries}) = {}^2C_2\left(\frac{7}{18}\right)^2\left(\frac{11}{18}\right)^0$$

$$= 1 \times \left(\frac{7}{18}\right)^2 = \left(7/18\right)^2$$

Q.5] Given:-

S_1 - Supply chain Level 1

S_2 - Supply chain Level 2

S_3 - Supply chain Level 3

$P(S_1) = \cancel{0.4}\ 40\% = 0.4$

$P(S_2) = 50\% = 0.5$

$P(S_3) = 1-[P(S_1)+P(S_2)] = 1-0.9 = 0.1$

$P(L/S_1) = 20\% = 0.2$	$P(L/S_1)$ - Late package by level 1
$P(L/S_2) = 40\% = 0.4$	(L/S_2) - Late package by level 2
$P(L/S_3) = 10\% = 0.1$	(L/S_3) = Late package level 3

To find:- $P\left(\frac{S_2}{L}\right) = ?$

∴ By Bayes Theorem

$$P\left(\frac{S_2}{L}\right) = \frac{P\left(\frac{L}{S_2}\right)\cdot P(S_2)}{\Sigma\left(\frac{L}{S_i}\right)\cdot(S_i)} = \frac{P\left(\frac{L}{S_2}\right)\cdot P(S_2)}{P\left(\frac{L}{S_1}\right)\cdot P(S_1)+P\left(\frac{L}{S_2}\right)\cdot P(S_2)+P\left(\frac{L}{S_3}\right)\cdot P(S_3)}$$

$$= \frac{0.4\times0.5}{(0.2\times0.4)+(0.4\times0.5)+(0.1)(0.1)}$$

$$= \frac{0.2}{0.29}$$

$$= 0.689655$$

Q.6.] If $X \sim U(1,2)$ and

$Y = \frac{3}{6-X}$ determine the PDF of Y.

$$f_X(x) = \frac{1}{b-a} = \frac{1}{2-1} = \frac{1}{1} = 1$$

$$Y = \frac{3}{6-X}$$

$$\therefore 6 - X = \frac{3}{Y}$$

$$\therefore 6 - \frac{3}{Y} = X.$$

$$\therefore X = 6 - \frac{3}{Y}$$

Differentiate with Y on both sides

$$\therefore \frac{dx}{dy} = -3 \ln Y.$$

$$F_y(y) = \cancel{f_X[g^{-1}(y)]} \; P(Y<y) = P(X < g^{-1}(y)) = P(X < 6 - 3/y)$$

$$\therefore f_y(y) = f_X(x) \cdot \left| \frac{d(g^{-1}(y))}{dx} \right|$$

$$= 1 \cdot |-3 \cdot \ln y|$$

$$= 3 \cdot \ln Y. //$$

Q.7]

X	1	2	3	4	5
$P(X=x)$	0.05	0.15	0.35	0.4	0.05

i) $F(3) = P(X=1) + P(X=2) + P(X=3)$

$= 0.05 + 0.15 + 0.35$

$= 0.55$

ii) $E(X) = \Sigma x \cdot P(X=x)$

$= 1\times0.05 + 2\times0.15 + 3\times0.35 + 4\times0.4 + 5\times0.05$

$= 3.25$

iii) $V(X) = E(X^2) - [E(X)]^2$

$\therefore E(X^2) = \Sigma x^2 \cdot P(X=x)$

$= 1^2\times0.05 + 2^2\times0.15 + 3^2\times0.35 + 4^2\times0.4 + 5^2\times0.05$

$= 0.05 + 0.6 + 3.15 + 6.4 + 1.25$

$= 11.45$

$\therefore V(X) = 11.45 - (3.25)^2$

$= 11.45 - 10.5625$

$= 0.8875$

iv) Coefficient of skewness.

$$\text{Coefficient of skewness} = \frac{3(\text{Mean} - \text{Median})}{\text{Standard deviation}}$$

Median of the given probability distribution is 3 because $P(X \leq 3) = 0.05 + 0.15 + 0.35 = 0.55$ which is greater than 0.5 and $P(X \geq 3) = 0.35 + 0.4 + 0.05 = 0.8$ which is greater than 0.5

Mean $= E(X) = 3.25$

$\therefore$ Median $= 3$

Standard deviation $= \sqrt{V(X)} = \sqrt{0.8875} = 0.942072|$

$\therefore$ Coefficient of skewness $= \dfrac{3(3.25-3)}{0.942072184}$

$= 0.7961173$ //

Q.8.] Insurance claim amounts are independent and exceed 1000 with probability of 0.2. In a sample of 15 claims, calculate the probability that fewer than 3 of them exceed 1000.

X – Insurance claims that exceed 1000.

Let them exceeding 1000 be a success and not exceeding 1000 be a failure.

$\therefore$ $p \neq$ p – Probability of success

q – probability of failure.

$\therefore$ $p+q=1$

$p = 0.2$ – given

$q = 1-0.2$

$= 0.8$

A sample of 15 claims is considered.

$\therefore$ $X \sim Bin(15, 0.2)$

We have to find probability that fewer than 3 of the claims exceed 1000.

$\therefore P(X<3) = P(X=0)+P(X=1)+P(X=2)$.

$P(X=x) = {}^{n}C_{x}\, p^{x} \cdot q^{n-x}$ — P.D.F of Binomial distribution.

$$\therefore P(X<3) = {}^{15}C_0 (0.2)^0 (0.8)^{15} + {}^{15}C_1 (0.2)(0.8)^{14} + {}^{15}C_2 (0.2)^2 (0.8)^{13}$$

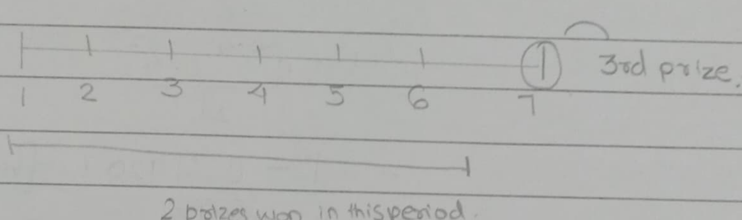
$$= \frac{15!}{15!\,0!} \times 1 \times (0.8)^{15} + \frac{15!}{14!\,1!} (0.2)(0.8)^{14} + \frac{15!}{13!\,2!} (0.2)^2 (0.8)^{13}$$

$$= (0.8)^{15} + \overset{15}{\cancel{1}} \times (0.2)(0.8)^{14} + \frac{15 \times \overset{7}{\cancel{14}}}{\cancel{2}} (0.2)^2 (0.8)^{13}$$

$$= 0.035184372 + \overset{0.1319413}{\cancel{0.0...}} + 0.2308974$$

$$= \cancel{0.2748778}\ 0.398023 \,//$$

Q.9.] A man enters a raffle each week where the probability of winning a price is 0.01. Calculate the probability that he takes 7 weeks up to and including the week where he wins his 3rd prize.



∴ X – ~~Prize~~ No. of. Prizes

∴ X follows Negative Binomial distribution with the parameters 7 and 0.01

$$\therefore P(X=x) = {}^{n-1}C_{x-1} \cdot p^{x} \cdot q^{n-x}$$

$$\therefore P(X=3) = {}^{7-1}C_{3-1} \cdot (0.01)^3 \cdot (0.99)^4$$

$$= {}^6C_2 \cdot (0.01)^3 (0.99)^4$$

$$= \frac{6!}{4!\,2!} (0.01)^3 (0.99)^4$$

$$= \frac{\overset{3}{\cancel{6}} \times 5}{\cancel{2}} (0.01)^3 (0.99)^4$$

$$= 15 \times (0.01)^3 \times (0.99)^4$$

$$= 0.000014409.$$

∴ The probability that it takes a man 7 weeks upto and including the week where he wins his 3rd prize is 0.000014409 //

Q. 10] The number of insurance claims submitted to an insurance office averages two per working week (of 5 days). Calculate the probability that more than two claims arrive in one day.

Let X - No. of claims

λ = Average no. of claims per week = 2.

$\therefore$ Average no. of claims per day $= \frac{2}{5} = 0.4$.

[$\because$ working week is of 5 days]

$\therefore X \sim P(0.4)$

$$P(X > 2) = 1 - P(X \leq 2)$$

$$= 1 - \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= 1 - \frac{e^{-0.4} \cdot 0.4^{2}}{2!}$$

$$= 1 - 0.99207 \text{ -[from table book]}$$

$$= 0.00793.$$

$\therefore$ The probability that more than two claims arrive in one day is 0.00793 //

Q.11] If the probability of having a male and female child is equal and a woman has 5 boys and two girls, calculate the probability that her next three children are boys.

M = Male child F = female child.

$P(M) = \frac{1}{2}$ $P(F) = \frac{1}{2}$

Let the birth of a boy be a sucess and of a girl be a failure.

$\therefore$ X — A Boy is born

$\therefore X \sim Bin(3, 1/2)$

$$\therefore P(X = x) = {}^{n}C_{x} \cdot p^{x} \cdot q^{n-x}$$

$\therefore P(X=3) = {}^3C_3 \; p^3 \cdot q^0$

$= \dfrac{3!}{3!\,0!} \left(\dfrac{1}{2}\right)^3 \left(\dfrac{1}{2}\right)^0$

$= 1 \times \left(\dfrac{1}{2}\right)^3$

$= \dfrac{1}{8}$

$= 0.125$

Q.12] Claims follow poisson process with rate 10 per day. Calculate the probability that the time between two claims is less than 0.1 days.

$\therefore X$ – No. of Claims

$\therefore X \sim Poi(10)$

Let one claim happen at time 0 and the next claim happens in less than 0.1 days.

D – Time of the next claim.

$\therefore D \sim Exp(10)$

$\therefore P(\cancel{X}D < 0.1) = 1 - e^{-\lambda x}$

$= 1 - e^{-10 \times 0.1}$

$= 1 - 0.367879$

$= 0.632121$

Q.13.] If X is following uniform distribution with parameters a = 75 and b = 120, Calculate the probability that X lies between 80 and 100

Given :- $X \sim U(75, 120)$

To find :- $P(80 < X < 100)$

$P(80 < X\ 100) = ?$

$P(X = x) = \frac{1}{b-a}$ — P.D.F for the continuous uniform distribution.

$F_X(x) = \frac{x-a}{b-a}$ — Cumulative probability of a continuous uniform distribution

$\therefore P(80 < x < 100) = P(X \leq 100) - P(X \leq 80)$

$= \left[\frac{100-75}{120-75}\right] - \left[\frac{80-75}{120-75}\right]$

$= \frac{25}{45} - \frac{5}{45}$

$= \frac{20}{45}$

$= 0.4445.$

Q.14.] If $X \sim Gamma(5, 0.4)$

Calculate $P(X < 22.89)$

$X \sim Gamma(5, 0.4)$

$\alpha = 5, \quad \lambda = 0.4$

$2\lambda X \sim \chi^2_{\mu = 2\alpha}$

$\therefore 2\lambda = 2 \times 0.4 = 0.8$

$\therefore$

$P(X < 22.89) = P(0.8X < 18.312)$

$= P(\chi^2_{\mu=10} < 18.312)$

By linear interpolation

$$0.5\left\{\begin{array}{l} P(\chi^2_{10} < 18.0) = 0.9648 \\ P(\chi^2_{10} < 18.5) = 0.9702 \end{array}\right\} 0.0054$$

— from table Book.

$\therefore 0.5 - 0.0054$

i.e for an increase in 0.5 the probability increase by 0.0054

$\therefore$ Dividing both sides by 0.5

$$\therefore 1 - \frac{0.0054}{0.5}$$

Multiplying both sides by 0.312

$$\therefore 0.312 - \frac{0.0054 \times 0.312}{0.5}$$

$$= 0.00336 96.$$

$$\therefore P(X < 22.89) = P(\chi^2_{10} < 18.312)$$

$$P(X < 22.89) = 0.0033696 \quad //$$

Q.15.) A continuous random variable has the following probability density function.

$$\frac{K}{(x+5)^4} \quad x > 0$$

$$\int_{-\infty}^{\infty} f(x).dx = 1.$$

$$\therefore \int_{-\infty}^{0} f(x).dx + \int_{0}^{\infty} f(x).dx = 1$$

$$\therefore 0 + \int_{0}^{\infty} \frac{K}{(x+5)^4} . dx = 1$$

$$\therefore K \int_{0}^{\infty} (x+5)^{-4} = 1$$

$$\therefore K\left[\frac{(x+5)^{-3}}{-3}\right]_0^{\infty} = 1$$

$$K\left\{\left[\frac{(\infty+5)^{-3}}{-3}\right] - \left[\frac{(0+5)^{-3}}{-3}\right]\right\} = 1$$

$$\therefore K\left(\frac{0}{-3} + \frac{(5)^{-3}}{3}\right) = 1$$

$$\therefore K \quad 0 + \frac{1}{(5)^3 \times 3} = 1$$

$$\therefore K\left(\frac{1}{125 \times 3}\right) = 1$$

$$\therefore K = 375$$

$$\text{ii) } F(10) = \int_0^{10} \frac{375}{(x+5)^4}$$

$$= 375 \int_0^{10} (x+5)^{-4}$$

$$= 375 \left[\frac{(x+5)^{-3}}{-3}\right]_0^{10}$$

$$= 375 \left[\frac{(10+5)^{-3}}{-3} - \frac{(0+5)^{-3}}{-3}\right]$$

$$= 375 \left[\frac{1}{(15)^3 \times -3} - \frac{1}{(5)^3 \times -3}\right]$$

$$= 375 \left[\frac{1}{3375 \times -3} - \frac{1}{125 \times -3}\right]$$

$$= 375 \times 0.002567902$$

$$= 0.9629632$$

$$\text{iii) } E(x) = \int_0^{\infty} x \cdot f_x(x) \cdot dx$$

$$\therefore E(x) = \int_0^\infty x \cdot \frac{375}{(x+5)^4} \cdot dx$$

$$= 375 \int_0^\infty \frac{\cancel{3}\, x}{(x+5)^4} \cdot dx$$

$$= 375 \int_0^\infty \frac{(x+5) - 5}{(x+5)^4} \cdot dx$$

$$= 375 \int_0^\infty \frac{x+5}{(x+5)^4} dx - \int_0^\infty \frac{5}{(x+5)^4} \cdot dx$$

$$= 375 \int_0^\infty (x+5)^{-3} \cdot dx - 5 \int_0^\infty (x+5)^{-4} \cdot dx$$

$$= 375 \left\{ \left[\frac{(x+5)^{-2}}{-2} \right]_0^\infty - 5 \left[\frac{(x+5)^{-3}}{-3} \right]_0^\infty \right\}$$

$$= 375 \left[\frac{(\infty+5)^{-2}}{-2} - \frac{(0+5)^{-2}}{(-2)} \right] - 5 \left[\frac{(\infty+5)^{-3}}{-3} - \frac{(0+5)^{-3}}{(-3)} \right]$$

$$= 375 \left\{ \left[0 + \frac{1}{5^2 \times 2} \right] - 5 \left[0 + \frac{1}{5^3 \times 3} \right] \right\}$$

$$= 375 \left(\frac{1}{50} \right) - \cancel{5} \times \frac{1}{\underset{25}{12\cancel{5}} \times 3}$$

$$= 375 \left[\frac{1}{50} - \frac{1}{75} \right]$$

$$= 375 \times 0.00666667$$

$$= 2.5$$

$$E(x) = 2.5$$