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Probability Assignment

1. $X \sim Pois(\theta)$ $E(X) = \theta$ $V(\theta) = \theta$

$\cancel{P[x \le 200]} = P[x \le 200.5]$

$\rightarrow \cancel{P\left[\frac{x-\theta}{\sqrt{\theta}} \le \frac{200.5-\theta}{\sqrt{\theta}}\right]} \rightarrow \sim N(0,1)$

$\Rightarrow P[x = 200] \rightarrow P[199.5 < x < 200.5] = \theta.$

$\rightarrow P[-1.96 < X < 1.96]$

$\rightarrow P[x_1 < \theta < x_2] = 0.95.$

CI: $\hat{\theta} = \theta + 1.96 \times \sqrt{\theta} = 200 + 1.96\sqrt{200}$

$= 200 + 1.96 \times 10\sqrt{2} = 200 \pm 27.7185$

$\rightarrow x_1 = \theta \pm 1.96\sqrt{\theta}$ CI: $\boxed{172.28, 227.7185}$

2. $X_i \rightarrow i = 1, 2, 3, 4 \dots n$ $E(X_i) = \mu$ $V(X_i) = \sigma^2$

i) $\rightarrow \bar{X} = \frac{\Sigma X_i}{n}$, $E(\bar{X}) = E\left(\frac{\Sigma X_i}{n}\right) = \frac{\Sigma E(X_i)}{n}$

$= \frac{\Sigma \mu}{n} = \frac{n \times \mu}{n} = \underline{\mu = E(\bar{X})}$

$\rightarrow V(\bar{X}) = V\left(\frac{\Sigma X_i}{n}\right) = \frac{\Sigma V(X_i)}{n^2} = \frac{\Sigma \sigma^2}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

ii. $E(S^2) = E\left[\left(\frac{1}{n-1}\right)\Sigma(X_i - \bar{X})^2\right]$

$= E\left[\left(\frac{1}{n-1}\right)\Sigma\left[(X_i)^2 + (\bar{X})^2 - 2X_i \times \bar{X}\right]\right]$

$= \cancel{E\left(\frac{1}{n-1}\right) \cdot \Sigma(X_i)^2 + \Sigma}$

$= \left(\frac{1}{n-1}\right) \times \Sigma\left[E(X_i^2) + E(\bar{X})^2 - 2(\bar{X}\bar{X})\right]$ $2E(\bar{X})$

$= (1/n-1) \times \Sigma\left[E(X_i^2) - E(\bar{X}^2)\right]$

$= (1/n-1) \times \cancel{\Sigma E(Y}$

$E(X_i^2) = V(X_i) + [E(X_i)]^2$ | $E(\bar{X}^2) = V(\bar{X}) + [E(\bar{X})]^2$

$= \sigma^2 + (\mu)^2$ | $= \sigma^2/n + \mu^2$

$= (1/n-1) \sum(\sigma^2 + \mu^2) - n[\sigma^2/n + \mu^2]$

$= (1/n-1)\ (n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2})$

$E(s^2) = \sigma^2$

$\rightarrow X_i \sim N(\mu, \sigma^2)$

$\rightarrow V(s^2).\quad \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$

$V\left[\frac{(n-1) \cdot s^2}{\sigma^2}\right] \sim V(\chi^2_{n-1}) = 2(n-1)$

$\frac{(n-1)^{\cancel{2}}}{\sigma^4} \times V(s^2) = 2\cancel{(n-1)}$

$\therefore V(s^2) = \frac{2\sigma^4}{(n-1)}$

iv) Pf $E(s^2) = \frac{\cancel{(n-1)}}{\sigma^2} E(s^2) = \cancel{(n-1)}$

$\therefore E(s^2) = \sigma^2$

$\rightarrow P[x_1 < \sigma^2 < x_2] = 0.50$

$\cancel{Z}.\quad \cancel{\alpha} = ' \quad CI = (1 - 50\%) = 0.50$

$\rightarrow Z_{25\%} \rightarrow 0.6745$

★ $\cancel{\rightarrow P\left[\frac{1}{x_1} < \frac{1}{\sigma^2} < \frac{1}{x_2}\right] = 0.50}$

$\cancel{\rightarrow P\left[\frac{(n-1)s^2}{x_1} < \frac{(n-1)s^2}{\sigma^2} < \frac{(n-1)s^2}{x_2}\right] = 0.50}$

$\cancel{\rightarrow \frac{(n-1)s^2}{x_1} = \chi^2_{n-1, 25\%}} \quad \Big| \quad \cancel{\frac{(n-1)s^2}{x_2} = \chi^2_{n-1, 75\%}}$

$\cancel{\rightarrow \therefore x_1 = \frac{9 \times}{\chi^2_{9, 25\%}}} \quad \cancel{x_1 = \frac{9}{(8.343)} \times}$

$\rightarrow \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ $\quad$ $\cancel{s^2 = \frac{\chi^2_{9, 50\%} \times (100)}{9} = \underline{92.7}}$

P[

iv.) $\sigma^2 = 100$

$E(s^2) = \sigma^2 = 100$

$\rightarrow P[50 < E(s^2) < 150] = P[\cancel{E}(s^2) < 150] - P[\cancel{E}(s^2) < 5$

$$\rightarrow \cancel{p[\sigma^2 < 150] - p[\sigma^2 < 50]}$$

$$= p\left[\frac{9 \times S^2}{100} < \frac{9 \times \cancel{150}^{3}}{\cancel{100}_{2}}\right] - p\left[\frac{9 \times S^2}{100} < \frac{\cancel{50}^{1} \times 9}{\cancel{100}_{2}}\right]$$

$$\rightarrow p\left[\chi^2_9 < 13.5\right] - p\left[\chi^2_9 < 4.5\right]$$

$$= 0.8587 - 0.1245$$

$$= 0.7342$$

$N = 4$

Q3 x: 0 1 2 3 4

x	0	1	2	3	4
$P(X=x)$	q^4	$p q^3 \cdot 4$	$6p^2q^2$	$4p^3q$	p^4
Freq:	86	75	16	2	1

$$L = \prod_{i=0}^{4} f(x_i) = (q^4)^{86} \cdot (p^3 \cdot q \cdot 4)^{75} \cdot (6p^2q^2)^{16} \cdot (4pq^3)^2 \cdot (p^4)^1$$

$$L = (1-p)^{344} \cdot (1-p)^{225} \cdot p^{75} \cdot p^{32} (1-p)^{32} \cdot p^{6} \cdot (1-p)^{2} \cdot p^4 \cdot \text{const}$$

$$\ln L = \ln\left[(1-p)^{603} \cdot p^{117} \cdot \text{const}\right]$$

$$\ln L = 603 \ln(1-p) + 117 \ln(p)$$ | Diwrt p $\rightarrow \frac{603(-1)}{1-p} + \frac{117}{p} = 0$

$\therefore \frac{117}{p} = \frac{603}{1-p}$ $\therefore \frac{1}{p} - 1 = \frac{603}{117}$ $\therefore 1/p = 6.15$

$\therefore \boxed{\hat{p} = 0.1625}$

ii.) Obs: 0 1 2 3 4

- Exp freq: $P(x=0) \cdot \text{Total}$: $(1-0.1625)^4 \times 180 = \underline{88.55}$
- $P(x=1) \cdot T \rightarrow 68.729$ $P(x=3) \rightarrow 2.587$

$P(x=2) \rightarrow 20$ $P(x=4) \rightarrow 0.1255$

$\therefore$ O:	86	75	16	2	1
E	88.55	68.72	~~2.587~~ 20	2.587	0.1255

$\frac{(O-E)^2}{E}$: 0.0734 0.573

As Exp freq of $x=3$ & 4 less than 5, we add it to 20.

$\therefore$ O	86	75	19
E	88.55	68.72	22.7125
$\frac{(O-E)^2}{E}$	0.0734	0.573	0.606

$= 1.2532$

$$\sum \frac{(O-E)^2}{E} \sim \chi^2_{3-1-1} = \chi^2_1 = 1.2532$$

$\rightarrow \chi^2_1 = 1.2532$, $\chi^2_1 \rightarrow$ at 5%: 3.841, at 10%: 2.706

As this is a fit test.

As $\chi^2_{cal} < \chi^2_{tab}$

Donot Reject H_0

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Q.4 $f(x) = \frac{c\beta^3}{(x+\beta)^4}$ $x>0, \beta>0$

i. $E(x) = \int x \cdot f(x)\, dx = \int_0^\infty x \cdot \frac{c\beta^3}{(x+\beta)^4}\, dx$

By looking at Pareto Distribution,

$\rightarrow f(x) = \frac{\alpha \lambda^\alpha}{(\lambda+x)^{\alpha+1}}$ ∴ $\lambda = \beta$, $\boxed{\alpha = 3, = c}$

$\frac{c\beta^3}{(x+\beta)^4}$ $c=3$

$\rightarrow \alpha = 3$, $\beta = \lambda$, $E(x) = \frac{\lambda}{\alpha - 1}$ $\boxed{\frac{\beta}{2} = E(x)}$

$V(x) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} \rightarrow \frac{3\times\beta^2}{(2)^2\cdot(1)} = \frac{3}{4}\times\beta^2 = Var(x)$

~~ii. By MOM: $E(X) = \bar{X}$~~

ii.) By MOM $\rightarrow E(x) = \bar{X}$

Random Sample $\rightarrow X_1\ X_2\ X_3 \dots X_N$

$\bar{X} = \frac{\sum X_i}{n} = \bar{X}_n$ 1, 2, 2, 4, 5 $\rightarrow U(1,5)$

$E(x) = \frac{\beta}{2}$: $\boxed{\hat{\beta} = 2\bar{X}_n}$ 3

$a = 3 - 2/\sqrt{3}\cdot\sqrt{3} = 1$

iii.) $E(\hat{\beta})$

$b = 3 + 2/\sqrt{3}\cdot\sqrt{3} = 5$

$= E(2\bar{X}_n)$ $s^2 = 4/3$

$= 2\sum \frac{E(X_i)}{n} = 2\times\frac{n\left(\frac{\beta}{2}\right)}{n} = \frac{\cancel{2}\times\cancel{n}\times\beta}{\cancel{2}\ \cancel{n}} = \hat{\beta} =$ Estimate

As $E(g(\theta)) = \theta$, it is unbiased.

$\rightarrow$ Consistency = MSE = Var + Bias^2 = $\boxed{\frac{3}{4}\times\beta^2}$ $= 0.75\beta^2$

for More Consistent

iii.) MSE of $(b\bar{X}_n) \rightarrow$ Estimator.

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Q5 $E(X) \sim Unif(a,b)$ $E(X) = \bar{X}$

$\frac{b+a}{2} = \bar{x}$ $\therefore a = 2\bar{x} - b$ $b = 2\bar{x} - a$.

$\frac{(b-a)^2}{12} = s^2$ $(2\bar{x} - 2a)^2 = 12s^2$

$2\bar{x} - 2a = 2\sqrt{3}s$

$\therefore \bar{x} - a = \sqrt{3}s$ $\therefore \boxed{\hat{a} = \bar{x} - \sqrt{3}s}$

ii) $\frac{(b - 2\bar{x} + b)^2}{12} = s^2$ $\therefore [2(b-\bar{x})]^2 = 12s^2$

$\cancel{2}(b-\bar{x}) = \cancel{2}\sqrt{3}s$

$\therefore b = \bar{x} + \sqrt{3}s$

iii) Sample: 1, 2, 3, 4, 5 $\longrightarrow U(1,5)$

$\bar{x} = \frac{a+b}{2} = \frac{5+1}{2} = 3$ | $s^2 = \frac{(b-a)^2}{12} = \frac{16}{12} = \frac{4}{3}$. $S = 2/\sqrt{3}$

$\rightarrow a = \bar{x} - \sqrt{3}s = 3 - \sqrt{\cancel{3}} \times 2/\sqrt{\cancel{3}} = 1$ $S = 1.581$

$b = \bar{x} + \sqrt{3}s = 3 + \sqrt{\cancel{3}} \times 2/\sqrt{\cancel{3}} = 5$

$\rightarrow a = 3 - \sqrt{3} \times (1.581) = 0.2616$

$b = 3 + \sqrt{3}(1.581) = 5.7383$

$\longrightarrow$ As a and B values are not same, ~~H~~OH is weak.

iv. PDF of Conti: $\frac{1}{b-a}$ $MLE = \prod_{i=1}^{n} f(x_i)$

~~$L = \left(\frac{1}{b-a}\right)^N = \left(\frac{1}{b}\right)^N$ $\ln L = N \cdot \ln\left(\frac{1}{b}\right)$~~

~~$\frac{d \ln L}{db} = \frac{N}{(1/b)} \times \frac{-1}{(b)^2} = \frac{b \times N(-1)}{b^2} = -\frac{N}{b} = 0$~~

~~$\rightarrow \ln L = N \cdot \ln\left[\frac{1}{b-a}\right] = \frac{d \ln L}{d(b)} = \frac{N}{(1/b-a)} \times \frac{-1}{(b-a)^2}$~~

$\rightarrow$ when $U(0,b)$, $E(x) = \frac{1}{b-a} = \frac{1}{b}$

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Q6 $H_0 = p = 0.1 \qquad H_1 : p > 0.1$

i) $n = 12 \rightarrow$ first step: If 3 or more hits i.e.
- Hits: If $p \geq 0.25$, reject Ho

$\rightarrow P[N \geq 3 / p = 0.1] = \cancel{10} \quad N \sim Bin(12, 0.1)$

$\rightarrow 1 - P[N < 3 / p = 0.1] = 1 - P[N = 0, N = 1, N = 2 / p = 0.1]$

$= 1 - [{}^{12}C_0 \cdot (0.1)^0 \cdot (0.9)^{12} + {}^{12}C_1 \cdot (0.1)^1 (0.9)^{11} + {}^{12}C_2 (0.1)^2 (0.9)^{10}]$

$1 - (0.282429 + 0.3765 + 0.2301) = \underline{0.1109}$

$\rightarrow n = 24, \quad 1 - P[N = 0, N = 1, N = 2, N = 3, N = 4 / p = 0.1]$

iii) $\cancel{P[N >}$ $P[N < 5 / p = 0.3]$

$= P[N = 0, N = 1, N = 2, N = 3, N = 4 / p = 0.3]$

$\rightarrow {}^{24}C_0 \cdot (0.7)^{24} + {}^{24}C_1 \cdot 0.3 \cdot (0.7)^{23} + {}^{24}C_2 \cdot (0.3)^2 \cdot (0.7)^{22} + {}^{24}C_3 \cdot (0.3)^3 \cdot (0.7)^{21} + {}^{24}C_4 \cdot (0.3)^4 \cdot (0.7)^{20}]$

$\rightarrow 0.0001915 + 0.00197 + 0.009711 + 0.030 + 0.0686$

$= \underline{0.11055}$

iv) $p = ?$ $Bin(1\cancel{0}, 1/3) \doteq N\left(p, \dfrac{(1-p)(p)}{n}\right)$

$\rightarrow x_1 \rightarrow$

Lower Bound - ? $Z_{5\%} \rightarrow \quad 1/3 \pm 1.6449 \times (0.14907)$

$\therefore 0.333 + 0.245, \; 0.333 - 0.245$

$\rightarrow 0.578, \; 0.0877$

v.) TS: $\dfrac{0.33 - 0.278}{\sqrt{\dfrac{0.33(0.667)}{10}}} = 0.36895 \rightarrow Z_{10\%} = \cancel{1.6449}$ 1.2816

As $Z_{cal} < Z_{tab}$,
we DN Reject Ho

As TS lied in the CI, we accepted Ho

vii) $\mu_A = \mu_B = H_0 \qquad \mu_A \neq \mu_B = H_1 \qquad \delta = \mu_A - \mu_B = 0$

TS: $\dfrac{(\mu_A - \mu_B) - \delta}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \sim N(0,1) \rightarrow \dfrac{(70-65) - 0}{\sqrt{\dfrac{54^2}{12} + \dfrac{70^2}{15}}}$

$\rightarrow \dfrac{5}{\sqrt{243 + 326.6}} = 0.2094 \rightarrow$ At 2.5% $\rightarrow$ 1.96
At 5% $\rightarrow$ 1.6449

As $Z_{cal} < Z_{tab}$, we DNR Ho for 5% LOS.

Assum $\rightarrow$ This SD is popⁿ SD.

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~~Q6~~ Q.7

i) Public → $\bar{x} = 30$ $\sigma^2 = 57.16667$

Private : $\bar{x} = 35.05$ $\sigma^2 = 67.4027$

ii.)

Q8: i.) Unbias Estimator : Difference between expected value of estimator and estimator is 0.

$E(\theta) = \theta$ If $\hat{\theta}$ is used to estimate $\hat{\theta}$, then if both are same, then $\hat{\theta}$ is an unbiased estimator

ii) Bias : when θ, used to estimate $\hat{\theta}$, doesnot give value, same as θ, then is called bias.

iii) MSE : Variance of Estimator wrt. true parameter value. MSE = Variance + Bias^2

iv) As MSE of $\tilde{\theta}$ is less, it is more consistent.
MSE is variance of estimator. if estimator is less then values are more consistent.

v) They would be termed consistent, when MSE ↓ or gradually becomes zero.

vi) Method of Moments of Method of Likelihood Estimator.

Q.9 i) In real life, population variance is never known, so we use S' as sample variance.
If $Z \sim N(0,1)$ and $C \sim \chi^2_k$, then $\dfrac{Z}{\sqrt{C/k}} \sim t_k$

i.e. the ratio of Standard Normal Distribution (z) and χ^2 distribution with 'k' degree of freedom (C)'s root follows 't' distribution with 'k' degree of freedom.

ii) when $k>2$, for 't': Mean $= 0$
Variance $= \dfrac{k}{k-2}$

iii) $n=10$ $\mu = 50$ $S = 48.667$ | $\alpha = 1\%$ $\therefore \alpha/2 = 0.5\%$

a.) $CI = \bar{x} \pm t_{\alpha/2, n-1} \times \left(\dfrac{S}{\sqrt{n}}\right)$

$= 50 \pm t_{0.5\%, 9} \times \dfrac{\sqrt{48.667}}{\sqrt{10}}$

$x_1 = 50 + 3.250 \times \cancel{15.3898}^{2.206} = \cancel{100.017}$ 57.1696

$x_2 = 50 - 3.250 \times \cancel{15.3898}$ 2.206 = 42.8305

→ CI for 99% is 42.8305 & 57.1696.

b.) Mean $= 0$ Var $= k/k-2$, $k=9$ $t_k = t_{n-1}$
Var $= 9/9-2 = 1.285714$.
→ $P\left[x_1 < \bar{x} < x_2\right] = 0.99$.

$P\left[\dfrac{x_1-\mu}{\sigma/\sqrt{n}} < z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} < \dfrac{x_2-\mu}{\sigma/\sqrt{n}}\right] = 0.99$

$Z_{\alpha/2} = Z_{0.5\%}$

$\therefore x_1 = \mu \pm 2.5758 \times \sqrt{\dfrac{\sigma^2}{n}}$

$\boxed{x_2 = 0.33117}$

$x_2 = \mu - 2.5758 \times \sqrt{\dfrac{\sigma^2}{n}}$

$\boxed{x_1 = -0.33117}$

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Q.11 $n=12$ $\Sigma x = 218200$ $\Sigma x^2 = 4919860000$

Sample Mean = 17,767 Sample ~~Var~~ SD = 10,144

i) Sample Mean = $\frac{\Sigma x}{N}$

ii) Sample SD = $\frac{\Sigma x^2 - n\cdot(\bar{x})^2}{n-1}$

Sample Mean = 17767 = $\frac{\Sigma x_s}{12}$, $\therefore \boxed{\Sigma x_s = 213204}$

New $\Sigma x_s = 213204 - 11000 - 48000 + 27500 + 31500$
$= 213204.$

New Sample Mean = $\frac{\text{New } \Sigma x_s}{N} = \frac{213204}{12} = \underline{17,767}$

ii) Sample SD = 10,144 = $\sqrt{\frac{\Sigma x_s^2 - n(\bar{x}_s)^2}{n-1}}$

$= (102900736) = \frac{(\overset{\Sigma x_s^2}{\cancel{4919860000}}) - 12\,[17767]^2}{11}$

$\rightarrow \therefore \Sigma x_s^2 = 4919903564$

New $\Sigma x_s^2 = 4919903564 - (11000)^2 - (48000)^2$
$+ (27500)^2 + (31500)^2$

New = 4243408564

$\therefore$ New SD = $\sqrt{\frac{\text{New } \Sigma x_s^2 - (n)\cdot(\text{New Mean})^2}{(n-1)}}$

~~Var~~$_s$ = $\sqrt{41400736}$

$\rightarrow$ New Sample SD = 6434.34037

* With change inputs the new sample mean remained constant, while new sample deviation reduced.

Q.12 i) CRLB is used when $n \to \infty$. Here in uniform distribution it ranges from 0 to θ, where θ is a parameter doesnot tend to infinity

ii) $y = \max X_i$ $\quad X_i \sim Unif(0, \theta)$ $\quad \therefore Y$ also follows $(0, \theta)$

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$\rightarrow$ Bias: $E(b\bar{X}_n) = b \times \frac{\sum E(X_i)}{n} = b \times \sum \times \frac{\beta}{2} \times \frac{1}{n}$

$= b \times \frac{\cancel{n} \times \beta}{2} \times \frac{1}{\cancel{n}} = \frac{b \times \beta}{2}$

$\rightarrow$ Bias: $b \times \frac{\beta}{2} - \beta = \frac{(b-2)\beta}{2}$: Bias

$Var(\beta) = V(b\bar{X}_n) = b^2 \times \frac{\sum V(X_i)}{n^2} = \frac{b^2 \times \cancel{n} \times \frac{3}{4} \times \beta^2}{n^2}$

$\rightarrow V(\beta) = \frac{3b^2\beta^2}{n \times 4}$ Bias $= \frac{(b-2)\beta}{2}$

MSE $=$ Var $-$ Bias^2 $= \frac{3}{4n} b^2\beta^2 - \frac{\beta^2}{4}[b^2+4-2b]$

$= \frac{3}{4n} b^2\beta^2 - \frac{\beta^2 b^2 + 4\beta^2 + 8b\beta^2}{4}$