Assignment - 1

Q1) $X \sim Poi(\theta)$ $\quad \mu = \theta$, $\sigma^2 = \theta$

$\sigma = \sqrt{\theta}$

$$\frac{X - \theta}{\sqrt{\theta}} \sim N(0,1)$$

$\hat{\theta} = 200$, $n = 1$

$$95\%\ CI = \mu \pm Z_{2.5\%}\, \sigma/\sqrt{n}$$
$$= 200 \pm 1.96\sqrt{200}$$
$$= 200 \pm 27.71858582$$
$$= (172.2814142,\ 227.71858582)$$

Q2) $X_i = 1, 2, \ldots n$ (iid)

$E(X_i) = \mu$

$V(X_i) = \sigma^2$

i) $\bar{x} = \dfrac{\sum_i X_i}{n}$

$$E(\bar{x}) = E\left(\frac{\sum X_i}{n}\right) = \frac{1}{n}\sum E(X_i) = \frac{1 \times n \times \mu}{n} = \mu$$

$$V(\bar{x}) = V\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2}\sum V(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

ii) $E(s^2) = \sigma^2$

$$s^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$$

ii) $$E(S^2) = E\left(\frac{1}{n-1}\sum(x_i-\bar{x})^2\right)$$

$$= \frac{1}{n-1} E\left(\sum(x_i-\bar{x})^2\right)$$

$$= \frac{1}{n-1} \cancel{E\left(\sum x_i^2\right)} \sum E(x_i)^2 - nE(\bar{x}^2)$$

$$= \frac{1}{n-1} \sum(\mu^2+\sigma^2) - n\left(\mu^2+\sigma^2/n\right)$$

$$\therefore E(x^2) = V(x) + \left[E(x)\right]^2$$

$$\therefore E(x_i^2) = V(x_i) + \left[E(x_i)\right]^2$$
$$= \sigma^2 + \mu^2$$

$$\therefore E(\bar{x}^2) = V(\bar{x}) + \left[E(\bar{x})\right]^2$$
$$= \frac{\sigma^2}{n} + \mu^2$$

$$= \frac{1}{n-1}\left(\cancel{n\mu^2} + n\sigma^2 - \cancel{n\mu^2} - \sigma^2\right)$$

$$= \frac{\cancel{(n-1)}(\sigma^2)}{\cancel{n-1}}$$

$$E(S^2) = \sigma^2$$

iii) $$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\therefore V\left[\frac{(n-1)S^2}{\sigma^2}\right] = V\left(\chi^2_{n-1}\right)$$

$$\therefore \frac{(n-1)^2 V(S^2)}{\sigma^4} = 2\cancel{(n-1)}$$

$$\therefore V(S^2) = \frac{2\sigma^4}{(n-1)}$$

iv) $x_i \sim N(\mu, \sigma^2)$

$\bar{x} \sim N(\mu, \sigma^2/n)$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$\sigma^2 = 100,\ n = 10$

$\therefore E(S^2) = \sigma^2 = 100$

$\therefore$ The values for $E(S^2)$ can be 50 to 150.

$P(S^2 \leq 50)$

$$P\left(\frac{(n-1)S^2}{\sigma^2} \leq \frac{50\times 9}{100}\right)$$

$$P\left(\chi^2_{n-1} \leq 4.5\right)$$

$$P\left(\chi^2_9 \leq 4.5\right)$$

$= 0.1245$

$P(S^2 < 150)$

$$P\left[\frac{(n-1)S^2}{\sigma^2} < \frac{150\times 9}{100}\right]$$

$$P\left(\chi^2_9 < 13.5\right)$$

$= 0.8587$

$\therefore P(50 < S^2 < 150) = 0.8587 - 0.1245$

$= 0.7342$

$$L = \prod_{i=1}^{n} {}^nC_x \; P$$

Q3)

No. of C	No. of P	E_i	$(O_i - E_i)^2/E_i$
0	86	88.55	0.0706
1	75	68.7	0.5777
2	16	20	0.8
3	2	2.6	
4	1	0.2	

$X \sim Bin(4, P)$

$$P(X=x) = {}^nC_x (P)^x (1-P)^{n-x}$$

$$L = \prod_{i=1}^{n} {}^nC_x (P)^x (1-P)^{n-x}$$

$$= P(x=0)^{86} \cdot P(X=1)^{75} \cdot P(X=2)^{16} \cdot P(X=3)^{2} \cdot P(X=4)^{1}$$

No. P	O_i	E_i	$(O_i - E_i)^2/E_i$
0	86	88.5	0.0706
1	75	68.7	0.5777
2+	19	22.8	0.6333
			1.2816

$$= \left({}^4C_0 (p)^0 (1-p)^4\right)^{86} \left({}^4C_1 (p)^1 (1-p)^3\right)^{75} \cdot \left({}^4C_2 (p)^2 (1-p)^2\right) \left({}^4C_3 (p)^3 (1-p)\right)^2 \left({}^4C_4 (p)^4 (1-p)^0\right)^1$$

$$\ln L = 86 \times 4 \log(1-P) + 75 \cdot \ln P + 225 \ln(1-P) + 32 \ln P + 32 \ln(1-P) + 6 \ln P + 2 \ln(1-P) + 4 \ln P$$

$$\frac{d \ln L}{dP} = \frac{-344}{1-P} + \frac{75}{P} - \frac{225}{1-P} + \frac{32}{P} - \frac{32}{1-P} + \frac{6}{P} - \frac{2}{1-P} + \frac{4}{P}$$

$$\therefore -\frac{603}{1-P} + \frac{117}{P} = 0$$

$$\therefore -603p + 117 - 117p = 0$$

$$\therefore 117 = 720p$$

$$\therefore \hat{p} = 0.1625$$

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$$\therefore P(X=0) = (1-p)^4 = 0.49197$$

$$\therefore P(X=1) = 4p(1-p)^3 = 0.38183$$

$$\therefore P(X=2) = 6p^2(1-p)^2 = 0.11113$$

$$\therefore P(X=3) = 4p^3(1-p) = 0.01437$$

$$\therefore P(X=4) = p^4 = 0.00070$$

$$\therefore \sum Z^2 = 1.2816$$

$$\therefore \chi^2_{1,5\%} = 3.841$$

df = n - 1 - no. of [illegible] - no. of [illegible]

$= 5 - 1 - 1 - 2$

$= 1$

$\therefore$ Since, $Z^2_{CAL} < Z^2_{TAB}$

$\therefore$ Do not reject H_0.

$\therefore$ The underlying data does follow binomial dist^n.

Q4) ~~$f(x) = \frac{C\beta^2}{(x+\beta)^4}$; $x>0$, $\beta>0$~~

Q5) $X \sim U(a,b)$

i) $\therefore \bar{x} = \frac{a+b}{2}$

$\therefore s^2 = \frac{(b-a)^2}{12}$

$\therefore s = \frac{b-a}{\sqrt{12}}$

$\therefore 2\sqrt{3} \cdot s = b - a$

$\therefore \hat{b} = \hat{a} + 2\sqrt{3}s$ — ①

$\therefore \bar{x} = \frac{(a+b)}{2}$

$\bar{x} = \frac{(a + a + 2\sqrt{3}s)}{2}$ — from ①

$\bar{x} = \frac{(2a + 2\sqrt{3}s)}{2}$

$\bar{x} = \hat{a} + \sqrt{3}s$

$\hat{a} = \bar{x} - \sqrt{3}s$

ii) $\hat{a} = \bar{x} - \sqrt{3}s$

~~$\hat{a} = \hat{b} + 2\sqrt{3}s$~~

$\therefore \hat{b} = \hat{a} + 2\sqrt{3}s$

~~$\hat{b} = \bar{x} - 3\sqrt{3}s$~~

$\hat{a} + 2\sqrt{3}s = \bar{x} - \sqrt{3}s + 2\sqrt{3}s$

$\therefore \hat{b} = \bar{x} + \sqrt{3}s$

iii) Sample $\rightarrow 1, 2, 3, 4, 20$

$\therefore \bar{x} = \frac{30}{5} = 6$

$\therefore s = 7.9057$

$\therefore \hat{a} = 6 - \sqrt{3}(7.9057)$
$= -7.6930$

$\therefore \hat{b} = 6 + \sqrt{3}(7.9057)$
$= 19.6930$

$\therefore$ The estimate values of $\hat{a}$ and $\hat{b}$ are not accurate.

iv) $X \sim U(0, b)$

$f(x) = \frac{1}{b-a} = \frac{1}{b}$

$L = f(x_1) \cdot f(x_2) \cdot f(x_3) \ldots f(x_n)$

$= \frac{1}{b} \cdot \frac{1}{b} \ldots \frac{1}{b}$

$= \frac{1}{b^n}$

$\ln L = -n \ln b$

$\frac{d \ln L}{db} = \frac{-n}{b}$

$\frac{d \ln L}{db} = \frac{-n}{b} = 0 \, , \; b \to \infty$

$\frac{d^2 \ln L}{db^2} = \frac{+n}{b^2} > 0$, min at $b \to \infty$

Q6) $H_0 : p = 0.1$
$H_1 : p > 0.1$

i) $P(\text{Type } 1) = P(\text{Reject } H_0 | H_0 \text{ is T})$

$= P(x \geq 3 | p = 0.1) + P(x = 0 | p = 0.1) \cdot P(Y \geq 5 | p = 0.1)$
$+ P(x = 1 | p = 0.1) \cdot P(Y \geq 4 | p = 0.1) +$
$P(x = 2 | p = 0.1) \cdot P(Y \geq 3 | p = 0.1)$
$= (1 - 0.8891) + (0.2824)(1 - 0.9957) +$
$(0.6590 - 0.2824)(1 - 0.9944) + (0.8891 - 0.6590)$
$(1 - 0.8891)$
$= 0.14727 \approx 15\%$

ii) $P(x \geq 3 | p = 0.3) + P(x = 0 | p = 0.3) \cdot P(Y \geq 5 | p = 0.3) +$
$P(x = 1 | p = 0.3) \, P(Y \geq 4 | p = 0.3) + P(x = 2 | p = 0.3)$
$P(Y \geq 3 | p = 0.3)$

$= 0.91253 \approx 91\%$

iii) $P(\text{Type II error}) = 1 - 0.91253 = 0.08747 \approx 9\%$

iii) ~~$\hat{a} = \bar{x} - \sqrt{3}s$~~

~~$\hat{b} = \hat{a} + 2\sqrt{3}s$~~

iv)

$X \sim Bin(120, 1/3)$ $\quad\quad \hat{p} = \frac{40}{120} = \frac{1}{3}$

$$\frac{X/n - p}{\sqrt{p(1-p)/n}} \sim N(0,1)$$

$$P\left(-1.2816 < \frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} < 1.2816\right)$$

$\therefore$ $(0.333, 0.2782)$ at 90% CI

v) $H_0 \Rightarrow p = 0.278$

$H_1 \Rightarrow p > 0.278$

$$\frac{X - np_0}{\sqrt{np_0(1-p_0)}} \sim N(0,1)$$

$$\frac{39.5 - 120 \times 0.278}{\sqrt{120 \times 0.278 \times 0.722}} = 1.291$$

$\therefore$ as that value falls in the C.I

$\therefore$ We do not reject H_0 at 10% L.o.S.

vii) $H_0 \Rightarrow \mu_1 = \mu_2$

$H_1 \Rightarrow \mu_1 \neq \mu_2$

$$\frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{1/n_1 + 1/n_2}} \sim \cancel{t_{n_1-1, n_2-1}}\ t_{n_1+n_2-2}$$

$$\frac{65 - 70 - 0}{63.459\sqrt{1/n + 1/s}} = 0.2034$$

t_{23} at 5% LS = $(-2.060, 2.060)$

Q7) Public $\Rightarrow$ Private $\Rightarrow$

i) $\bar{x}_1 = 30$ $\bar{x}_2 = 35.06$

$S_1^2 = 64.31$ $S_2^2 = 67.42$

ii) $H_0 \Rightarrow \sigma_1^2 = \sigma_2^2$

$H_1 \Rightarrow \sigma_1^2 \neq \sigma_2^2$ $S_p^2 = 65.8574$

$$\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1, n_2-1}$$

$$= \frac{64.31}{67.42} = 0.9542$$

$F_{8,8}$ at LoS 5% $= (0.2256, 4.433)$

ii) $H_0 \Rightarrow \mu_1 = \mu_2$, $H_1 \Rightarrow \mu_1 \neq \mu_2$

$$\frac{(30 - 35.06) - 0}{\sqrt{65.8574} \cdot \sqrt{2/9}} = 1.32$$

$t_{n_1+n_2-2} = t_{16}$ at 5% LoS $= (-2.12, 2.12)$.

$\therefore$ Do not reject H_0.

$\therefore$ The population means are equal at 5% LoS

Q8)

i) $\hat{\theta}$ is said to be unbiased when $E(\hat{\theta}) = \theta$

ii) Measure of bias is given by $E(\hat{\theta}) - \theta$

iii) M.SE of $\hat{\theta} = (E(\hat{\theta}) - \theta)^2$

iv) $\tilde{\theta}$ is a more efficient estimator because it will have a lower variance, while $\hat{\theta}$ having no bias has a bigger variance.

v) An estimator is said to be consistent, when MSE converges to zero as $n \to \infty$

vi) Method of Moments (MOM)
Maximum Likelihood Generator (MLE).

Q9)

i) $t_k = \dfrac{Z}{\sqrt{\chi_k^2/k}}$

$Z \sim N(0,1)$
χ_k^2 = chi-square dist^n with k d.o.f.
k = degree of freedom

ii) $E(t_k) = 0$ $\quad V(t_k) = \dfrac{k}{k-2}$ for $k > 2$

iii) $n = 10$
$\bar{x} = 50$
$S^2 = 48.667$

a) $\dfrac{\bar{x} - \mu}{S/\sqrt{n}} \sim t_{n-1}$, $t_9 = 3.25$

CI for 99% $= 50 \pm 3.25\sqrt{\dfrac{48.667}{10}}$
$= (42.83, 57.17)$

b) $t_{n-1} \sim N\left(0, \sqrt{\frac{n-1}{n-3}}\right) \sim N(0, \sqrt{9/7}$

$$\frac{\bar{X}-\mu}{S/\sqrt{n}} \sim N(0, 3/\sqrt{7})$$

$$\frac{\bar{x}-\mu}{S/\sqrt{7n/9}} \sim N(0,1)$$

$$C.I = 50 \pm 2.58 \sqrt{\frac{48.667}{10} \times \frac{9}{7}}$$

$$= (43.54, 56.45).$$

Q10) $n = 1000$
$X \sim Poi(3)$

$X \sim N(3000, 3000)$

i) Type 1 error $= P(H_0 \text{ is reject} \mid H_0 \text{ is true})$
$= P(X > 3100 \mid \lambda = 3)$
$= P\left(Z > \frac{3100 - 3000}{\sqrt{3000}}\right)$
$= P(Z > 1.8257)$
$= 1 - P(Z < 1.8257)$
$= 3.39\%$

ii) Type 2 error $= P(\text{Accepting } H_0 \mid H_0 \text{ is F})$.

iii) Power $= P(\text{reject } H_0 \mid H_0 \text{ is false})$
$= P(X > 3100 \mid \lambda = n\lambda)$
$= 1 - P\left(\frac{3100 - n\lambda}{\sqrt{n\lambda}}\right)$

iv) $\lambda = 2900$ $\quad X \sim N(2900, 2900)$

$$C.I\ (99\%) = \mu \pm 2.5758\sqrt{\frac{2900}{1000}}$$

$$= 2900 \pm 2.5758\sqrt{2.9}$$
$$= 2900 \pm 4.3864$$
$$= 2900 \pm 4.3864$$
$$= \cancel{2900 \cdot 3864}\ (2895.61, 2904.38)$$
$$= (2.8956, 2.9042)$$

Q1)

$\Sigma x = 213200$
$\Sigma x^2 = 4919860000$
$\bar{x} = 17767$
$S = 10144$

$$\Sigma x_1 = 213200 - 11000 - 48000 + 27500 + 31500$$
$$= 213200$$

$$\bar{x}_1 = \frac{\Sigma x_1}{n} = \frac{213200}{12} = \underline{\underline{17767}}$$

$$\Sigma x_1^2 = 4919860000 - 11000^2 - 48000^2 + 27500^2 + 31500^2$$
$$= 4243360000$$

$$S_1 = \sqrt{\frac{4243360000 - 12(17767)^2}{11}}$$
$$= 6438.0326$$

Q12) $x_1, x_2 \dots x_n$ $\quad x_i \sim U(0,\theta)$

$a=0,\ b=\theta$

i) $E(x_i) = \frac{a+b}{2}$

$= \frac{0+\theta}{2} = \frac{\theta}{2}$

$V(x_i) = \frac{(b-a)^2}{12}$

$= \frac{(\theta-0)^2}{12}$

$= \frac{\theta^2}{12}$

$L = \prod_{i=1}^{n} \left(\frac{1}{\theta}\right)$

$= \frac{1}{\theta^n}$

$\ln L = -n \cdot \ln\theta$

$\frac{d\ln L}{d\theta} = \frac{-n}{\theta}$

$\frac{d^2 \ln L}{d\theta^2} = \frac{+n}{\theta^2} > 0, \ \theta \to \text{min}.$

Q5) $f(x) = \frac{c\beta^3}{(x+\beta)^4}$, $x>0,\ \beta>0$

i) $\int_0^\infty f(x)\cdot dx = 1$

$\int_0^\infty \frac{c\beta^3}{(x+\beta)^4}\, dx = 1$

$c\beta^3 \left[\frac{\cancel{c\beta}\,(x+\beta)^{-3}}{-3}\right]_0^\infty = 1$

$\frac{c\beta^3}{-3}\left[0 - \beta^{-3}\right] = 1$

$c = 3$

i) $E(x) = \int_0^\infty x \cdot f(x) \cdot dx$

$= 3\beta^3 \int_0^\infty \frac{x}{(x+\beta)^4} \cdot dx$

$= 3\beta^3 \int_0^\infty \frac{x+\beta}{(x+\beta)^4} - \frac{\beta}{(x+\beta)^4} \cdot dx$

$= 3\beta^3 \left[\frac{(x+\beta)^{-2}}{-2} - \frac{\beta(x+\beta)^{-3}}{-3} \right]_0^\infty$

$= 3\beta^3 \left[0 - 0 - \left[\left(\frac{\beta^{-2}}{-2}\right) - \left(\frac{\beta^{-2}}{-3}\right) \right] \right]$

$= \frac{\beta}{2}$

ii) $x_1, x_2, x_3 \quad \ldots \quad x_n$

$E(x) = \bar{x} \qquad \hat{\beta} = 2\bar{x}_n$

$E(\hat{\beta}) = E(2\bar{x}_n) = 2 \times \frac{\beta}{2} = \beta \therefore$ unbiased.

iii) $MSE = Var(b\bar{x}_n) + [Bias(b\bar{x}_n)]^2$

$= b^2 \times \frac{3}{4} \times \frac{\beta^2}{x} + E(b\bar{x}_n) - \beta)^2$

$= \frac{\beta^2}{n} \left[\frac{3b^2}{4} + n\left(\frac{b}{2} - 1\right)^2 \right]$

iv) It sem in that $\hat{\beta} = 2\bar{x}_n$ is an unbiased estimator
It is also consistent.