Probability & Stats
Assignment 1.

1) $X \sim Poi(\theta)$

$$\frac{X - \theta}{\sqrt{\theta}} \mathrel{\dot\sim} N(0,1) \text{ follows } \frac{R.V - mean}{s.d} \text{ form.}$$

& The normal approximation follows.

$\hat{\theta} = 200$ $\quad \hat{\theta} = \frac{\Sigma x_i}{n}$.

~~aggregation~~ ~~$\hat{\theta}$~~ $x = \hat{\theta}$ since there is a single observation.

$$\frac{\hat{\theta} - \theta}{\sqrt{\theta/n}} \mathrel{\dot\sim} N(0,1).$$

$$P\left(Z_{2.5\%} < Z < Z_{97.5\%}\right) = 0.95.$$

$$P\left(-1.96 < \frac{\hat{\theta} - \theta}{\sqrt{\theta/n}} < 1.96\right) = 0.95.$$

$$95\% \text{ CI} = \bar{X} \pm Z_{\alpha/2}\sqrt{\cancel{X}/n}$$

$$= \hat{\theta} \pm Z_{\alpha/2}\sqrt{\theta/n}$$

$$= 200 \pm 1.96\sqrt{200/1}$$

$$= 200 \pm 1.96 \cdot 14.142$$

$$= 200 \pm 27.718$$

$$= (172.281, 227.718)$$

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$X_i : i = 1, 2, \dots n$ are n iid R.V's.

$\bar{x} = \sum_{i=1}^{n} x_i \qquad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$.

i) $E(\bar{x}) = E\left(\frac{\sum x_i}{n}\right)$

$\Rightarrow \frac{1}{n} E(\sum x_i) \Rightarrow \frac{1}{n} \sum E(x_i)$

$\Rightarrow \frac{1}{n} \sum \bar{x} \Rightarrow \cancel{\frac{1}{n} \frac{\mu}{n}}$

$\Rightarrow \frac{1}{n} . n\mu \Rightarrow \mu$.

$\therefore E(\bar{x}) = \mu$.

$V(\bar{x}) = V\left(\frac{\sum x_i}{n}\right) \Rightarrow \frac{1}{n^2} V(\sum x_i)$

$\frac{1}{n^2} \sum V(x_i) \Rightarrow \frac{1}{n^2} \sum \sigma^2$.

$\Rightarrow \frac{n\sigma^2}{n^2} \Rightarrow \frac{\sigma^2}{n}$.

$\therefore V(\bar{x}) = \frac{\sigma^2}{n}$.

ii) $E(s^2) = E\left[\frac{1}{n-1} \left[\sum x_i^2 - n\bar{x}^2\right)\right]$

$= \frac{1}{n-1} E(\sum x_i^2 - n\bar{x}^2)$

$\Rightarrow \frac{1}{n-1} \sum E(x_i)^2 - n E\bar{x}^2$

$$= \frac{1}{n-1}\left[\Sigma(\mu^2+\sigma^2) - n\left(\mu^2+\frac{\sigma^2}{n}\right)\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 + \cancel{n\mu^2} - \cancel{n\mu^2} - \sigma^2\right]$$

$$\Rightarrow \frac{1}{n-1}\left[n\sigma^2-\sigma^2\right] \Rightarrow \frac{1}{\cancel{n-1}}\sigma^2\cancel{(n-1)}$$

$$= \sigma^2.$$

$\therefore E(s^2) = \sigma^2$ proved

iii) $var\left[\frac{(n-1)s^2}{\sigma^2}\right] = var(\chi^2_{n-1}).$

$var(\chi^2_{n-1}) = 2(n-1)$ as $v(\chi^2_k) = 2k.$

$\therefore var\left[\frac{(n-1)s^2}{\sigma^2}\right] = 2(n-1)$

$\Rightarrow \frac{(n-1)^2}{\sigma^4} var(s^2) = 2(n-1)$

$\Rightarrow var(s^2) = \frac{2\cancel{(n-1)}\sigma^4}{\cancel{(n-1)^2}}$

$= v(s^2) = \frac{2\sigma^4}{(n-1)}$

proved

iv) $n = 10 \quad \sigma^2 = 100$

$E(s^2) = \sigma^2 = 100.$

$P[50 < s^2 < 150].$

$$\cancel{P[S^2 < 50]} \quad P[S^2 < 150] - P[S^2 < 50].$$

$$P\left[\frac{(n-1)s^2}{\sigma^2} < \frac{9(150)}{(10)^2}\right] - P\left[\frac{(n-1)s^2}{\sigma^2} < \frac{9(50)}{(10)^2}\right]$$

$$\Rightarrow \cancel{P\left[(n-1)s^2\right.}$$

$$P\left[\chi^2_9 < 13.5\right] - \cancel{P\left[\chi^2_9 < 4.5\right]}$$

$$P\left[\chi^2_9 < 13.5\right] - P\left[\chi^2_9 < 4.5\right]$$

$$\Rightarrow \cancel{(1 - 0.8587) - (1 - 0.1245)}$$
$$\cancel{= -0.7342}$$

$$= 0.8587 - 0.1245$$
$$= 0.7342.$$

3) $n = 4$

PF of binomial $= {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$

For $x = 0$ $\quad {}^{4}C_{0}\, p^{0} (1-p)^{4-0}$.

$= (1-p)^{4}$.

For $x = 1$ $= {}^{4}C_{1}\, p^{1} (1-p)^{3}$

$= 4p(1-p)^{3}$

Similarly $L = \prod_{i=1}^{4} {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$.

$L = (1-p)^{4} \cdot 4p(1-p)^{3} \cdot 6p^{2}(1-p)^{2} \cdot 4p^{3}(1-p) \cdot p^{4}$

$L = 96 p^{10} (1-p)^{10}$.

$\log L = \log 96 + 10 \log p + 10 \log (1-p)$

$\frac{d(\log L)}{dp} = \frac{10}{p} + \frac{10}{1-p}(-1)$

$= \frac{10}{p} - \frac{10}{1-p}$

$0 = \frac{10(1-p) - p}{p(1-p)}$

$\Rightarrow 0 = 10(1-p) - p$.

$0 = 10 - 10p - p$.

$0 = 10 - 11p$.

$p = \frac{10}{11} \Rightarrow \hat{p} = 0.9091$

$$\frac{d^2[\ln L]}{dp^2} = -\frac{10}{p^2} - \frac{10}{(1-p)^2}$$

which is < 0.

$\therefore$ function is maximum at

$\hat{p} = 0.9091$

ii)

x_i	O_i	E_i
0	86	122.95
1	75	49.17
2	16	7.37
3	2	0.2916
4	1	0.012

$$P(x=0) = {}^4C_0 \, p^x (1-p)^{n-x} \times 180$$

$$= 122.95$$

Similarly for $P(X=1)$, $P(x=2)$, $P(x=3)$, $P(x=4)$

For Goodness of fit

$$z^2 = \frac{(O_i - E_i)^2}{E_i}$$

$(O_i - E_i)^2 / E_i$

11.1045

13.569

10.195

$\cancel{9.628}$

$\cancel{8.345}$

15.723 → clubbing last 3 rows

40.3965

$Z_{cal} = 40.3965$

$Z_{tab} = \chi^2_{1, 5\%} = 3.841$

Since $Z_{cal} > Z_{tab}$, H_0 will be rejected.

4) $f(x) = \dfrac{c\beta^3}{(x+\beta)^4}$; $x > 0, \beta > 0$

$\int_0^\infty f(x)\,dx = 1 \Rightarrow \int_0^\infty \dfrac{c\beta^3}{(x+\beta)^4}\,dx = 1$

$1 = c\beta^3 \left[\dfrac{(x+\beta)^{-3}}{-3} \right]_0^\infty$

$= \dfrac{c\beta^3}{-3} \left[\dfrac{1}{(x+\beta)^3} \right]_0^\infty$

$$1 = c\beta^3 \left[\cancel{\frac{(x+\beta)^{-3}}{-3}}\right]_0^\infty$$

$$= \frac{c\beta^3}{-3}\left[0 - \frac{1}{\beta^3}\right] = \frac{c}{3}$$

$$\Rightarrow c = 3.$$

$f(x) = \dfrac{3\beta^3}{(x+\beta)^4}$ is a pareto distribution with $\alpha = 3$.

$$E(X) = \frac{\lambda}{\alpha - 1} \Rightarrow \frac{\beta}{\alpha - 1} = \frac{\beta}{2}.$$

$$V(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}$$

$$\Rightarrow \frac{3\beta^2}{2^2(3-2)} \Rightarrow \frac{3\beta^2}{4}$$

ii) $\bar{X} = \mu.$

$$\frac{\sum x_i}{n} = \mu = \frac{\beta}{2}.$$

$$\hat{\beta} = 2\frac{\sum x_i}{n} \Rightarrow 2\bar{x}_n$$

$$E(\hat{\beta}) - \beta = E(2\bar{x}_n) - \beta$$

$= 2E(\bar{X}_n) - \beta$

$= 2\left(\frac{\beta}{2}\right) - \beta$

$= 0$

$\therefore \hat{\beta}$ is an unbiased estimator of β

$MSE(\hat{\beta}) = Var(\hat{\beta}) + [Bias(\hat{\beta})]^2$

Since $Bias(\hat{\beta}) = 0$

$Var(\hat{\beta}) = Var(2\bar{X}_n)$

$= 4\,Var(\bar{X}_n) = 4\,Var\left(\frac{\Sigma X_i}{n}\right)$

$= \frac{4}{n^2} Var(\Sigma X_i)$

$= \frac{4}{n^2} \Sigma\, Var(X)$

$= \frac{4}{n^2}(n)\left(\frac{3}{4}\beta^2\right)$

$= \frac{3\beta^2}{n}.$

$MSE(\hat{\beta}) = \frac{3\beta^2}{n} + 0 = \frac{3\beta^2}{n}.$

As $n \to \infty$, β will $\to \infty$

$\therefore \hat{\beta}$ is consistent

iii $MSE = Var + Bias^2$

$$Bias(b\bar{x}_n) = E[b\bar{x}_n] - \beta$$
$$= bE(\bar{x}_n) - \beta$$
$$= b\left(\frac{\beta}{2}\right) - \beta$$
$$= \beta\left(\frac{b}{2} - 1\right)$$

$$Var(b\bar{x}_n) = b^2 Var(\bar{x}_n)$$
$$= \frac{b^2}{n^2} Var \Sigma \bar{x}_i$$
$$= \frac{b^2}{n^2} n \frac{3\beta^2}{4} = \frac{3\beta^2 b^2}{4n}$$

$$MSE(b\bar{x}_n) = \frac{3\beta^2 b^2}{4n} + \beta^2\left(\frac{b}{2} - 1\right)^2$$
$$\cancel{= 6b\beta^3 + \beta^2\left(\frac{b}{2} - 1\right)}$$

$$0 = \frac{6b\beta^2}{4n} + \beta^2\left(\frac{b}{2} - 1\right)$$

$$1 = \frac{3b}{2n} + \frac{b}{2}$$

$$1 = \frac{b}{2}\left[\frac{3}{n} + 1\right] \Rightarrow \frac{2}{1 + 3/n} = b$$

iv) According to (ii), $\hat{\beta} = 2\bar{x}_n$ is an unbiased estimator.

$b\bar{x}_n$ is negetively braised for $b \le 2$, $n \ge 1$

when $b = \dfrac{2}{1+3/n}$

Also, $n \to \infty$, $b \to 2$

∴ estimator is consistent

5 i) $x \sim U(a, b) \Rightarrow E(x) = \bar{x} = \dfrac{b+a}{2}$.

$2\bar{x} = b + a. \rightarrow (1)$

$Var(x) = \dfrac{(b-a)^2}{12} \Rightarrow S = \sqrt{\dfrac{(b-a)^2}{12}}$

$2\sqrt{3}\,S = b - a \rightarrow (2)$

Solving (1) and (2) simultaneously,

$2\sqrt{3}\,S + a + a = 2\bar{x}$

$\Rightarrow 2\sqrt{3}\,S + 2a = 2\bar{x}$

$\sqrt{3}\,S + a = \bar{x}$

$\therefore \hat{a} = \bar{x} - \sqrt{3}\,S$ proved

ii) For $\hat{b}$,

$b + (2\sqrt{3}\, b - 2\sqrt{3}S) = 2\bar{x}$

$\Rightarrow b + b - 2\sqrt{3}\,S = 2\bar{x}$

$2b - 2\sqrt{3}S = 2\bar{x}$

$b - \sqrt{3}S = \bar{x}$

$\hat{b} = \bar{x} + \sqrt{3}S$

iii) Let the sample of 5 observation be
1, 3, 5, 7, 70

$\bar{x} = 17.2 \qquad S = 29.601$

$\hat{a} = \bar{x} - \sqrt{3}S$

$\hat{a} = 17.2 - \sqrt{3}(29.601)$

$= -34.07$

$\hat{b} = \bar{x} + \sqrt{3}S$

$= 17.2 + \sqrt{3}(29.601)$

$= 68.47$

Potential weakness of a MOM estimator to estimate a and b of $U(a, b)$ is an invalid value. We can see that there is a value (70) in the sample which is greater than $\hat{b}$ which contradicts $U(a, b)$ and thus is invalid.

iv) PDF $= \frac{1}{b-a}$ but $a = 0$

$\therefore$ PDF $= \frac{1}{b}$

$$L = \prod_{i=1}^{n} \frac{1}{b} = \frac{1}{b^n}$$

$$0 \le x_1 < x_2 < \dots x_n \le b.$$

For L to be maximum, b^n should be minimum at b maximum.

$b = x_{max}$.

$\therefore$ MLE $= x_{max}$

6i) $H_0 : p = 0.1$ $\quad H_1 : p > 0.1$

$$P(\text{Type } 1) = P(x \geq 3) + P[x=2]\,P[x' \geq 3] + P[x=1]\,P[x' \geq 4] + P[x=0]\,P[x' \geq 5]$$

$$= (1 - 0.8891) + (0.2824)(0.0043) + (0.659 - 0.2824)(1 - 0.9744) + (0.8891 - 0.659)(1 - 0.8891) = 0.14727$$

$\approx 15\%$.

ii) $= P[x \geq 3] + P[x=2]\,P[x' \geq 3] + P[x=1]\,P[x \geq 4] + P[x=0]\,P[x \geq 5]$

$$= 0.7472 + (0.0138)(0.2763) + (0.085 - 0.0138)(0.5075) + (0.2528 - 0.085)(0.7472)$$

$= 0.91253 = 91.253\%$.

iii) $P[\text{type II error}] = P[\text{Accept } H_0 \mid H_0$ is

$= 1 - 0.91253$

$= 0.08747$

$= 8.747\%$

iv) $\dfrac{\hat{p} - p}{\sqrt{\dfrac{\hat{p}\hat{q}}{n}}} \sim N(0,1)$

$\hat{p} = \dfrac{40}{120} = \dfrac{1}{3}$

$\hat{q} = \dfrac{2}{3}$

$Z_{10\%} = 1.2816 \Rightarrow \hat{p} - 1.2816\sqrt{\dfrac{\hat{p}\hat{q}}{n}}$

$= \dfrac{1}{3} - 1.2816\sqrt{\dfrac{1/3 \times 2/3}{120}}$

$= 0.2782$

v) $H_0 : p = 0.278 \qquad H_1 : p > 0.278$

$z = \dfrac{x - np}{\sqrt{npq}} = \dfrac{39.5 - 120(0.278)}{\sqrt{120(0.278)(1-0.278)}}$

$= 1.2511$

$z_{tab} = 1.2816$

As $Z_{cal} < Z_{tab}$ we do not reject H_0 at LOS 10%.

vi) By the CI we know that there is a 10% chance of $p \leq 0.2782$

By the hypothesis test we know that p could be less than or equal to 0.278 with probability more than 10%. (approximately 10.5%)

vii) $H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$

$$\frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

$\bar{x}_1 = 65$ $\bar{x}_2 = 70$ $S_1 = 54$ $S_2 = 20$

$n_1 = 12$ $n_2 = 15$

$S_p^2 = 4027.04 \Rightarrow S_p = 63.459$

$$\frac{65-70}{63.459\sqrt{\frac{1}{12}+\frac{1}{15}}} = -0.2034$$

$t_{cal} = -0.2034$ $t_{tab} = \pm 2.060$

As t_{cal} lies between t_{tab} we do not reject H_0 at LOS 5%.

7. i) $\bar{x}_1 = \frac{\Sigma x_i}{n} = \frac{270}{9} = 30$

$\bar{x}_2 = \frac{\Sigma x_i}{n} = \frac{315.5}{9} = 35.055$

$S_1^2 = 64.3125$
$S_2^2 = 67.4027$

ii) The condition for testing equal mean is that the population variances should be equal. When we look at the sample variances, we see that both are sufficiently close that we assume that the population variances are the same.

iii) $H_0 : \mu_2 = \mu_1 \quad H_1 : \mu_2 > \mu_1$

$$\frac{(\hat{\mu}_2 - \hat{\mu}_1) - (\mu_2 - \mu_1)}{sp\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$\Rightarrow \frac{(\bar{x}_2 - \bar{x}_1) - (\mu_2 - \mu_1)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$S_p = \sqrt{\frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 1}}$$

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$$\sqrt{\frac{8(64.3125) + 8(67.4027)}{9 \cancel{8} + \cancel{8}9 - 1}}$$

$= \cancel{8.3814}\ 8.1153$

$$Z_{cal} = \frac{35.055 - 30}{8.3814\sqrt{2/9}}$$

$Z_{cal} = 1.321$

$Z_{tab} = 1.746$

As $Z_{cal} \cancel{\geq} < Z_{tab}$ we do not reject H_0 at LOS 5%.

8i) If Parameter $= \theta$ and Estimator $= \hat{\theta}$.

Unbiased estimator is when $E(\hat{\theta}) = \theta$

ii) $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$

iii) $MSE(\hat{\theta}) = Var(\hat{\theta}) + [Bias(\hat{\theta})]^2$
$= E(\hat{\theta} - \theta)^2$

iv) $Bias(\tilde{\theta}) = [E(\tilde{\theta}) - \theta]$ = +ve.
$\therefore E(\tilde{\theta}) > \theta$

$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta = 0$
$\therefore E(\hat{\theta}) = \theta$

$MSE(\hat{\theta}) > MSE(\tilde{\theta})$

For a parameter to be efficient MSE should be low. Therefore despite having greater bias than $\hat{\theta}$, $\tilde{\theta}$ is a better or more efficient estimator.

v) For consistency if by the increasing the sample size, if as $n \to \infty$, $MSE \to 0$ is true, then the estimator is consistent

vi) Two methods of estimating θ are
→ Method of moments (MOM)
→ Maximum Likelihood Estimator (MLE)

9 i) $t_k = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$

$\bar{x}$ = Sample mean
μ = Population mean
s = Sample standard deviation
n = number of observation
k = degrees of freedom.

ii) $E(t_k) = 0$ for all $k > 1$
$var(t_k) = \dfrac{k}{k-2}$ for all $k > 2$

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iia) $n = 10 \qquad \bar{x} = 50 \qquad s^2 = 48.667$

a) $t_k = \dfrac{\bar{x} - \mu}{s/\sqrt{n}} = \dfrac{50 - \mu}{\sqrt{48.667/10}}$

$= \dfrac{50 - \mu}{2.2061}$

$t_{0.005,9} \leq \dfrac{50 - \mu}{2.2061} \leq t_{0.05,9}$

$-3.25 \leq \dfrac{50 - \mu}{2.2061} \leq 3.25$

$\Rightarrow \quad -3.25(2.2061) \leq 50 - \mu \leq 3.25(2.2061)$

$50 + 3.25(2.2061) \geq \mu \geq 50 - 3.25(2.2061)$

$CI_{99\%} = (42.8802, 57.169)$

b) $t_{n-1} \sim N(0, \frac{9}{7})$

$\dfrac{\bar{x} - \mu}{s/\sqrt{n}} \sim N(0, 9/7)$

$\dfrac{\dfrac{\bar{x} - \mu}{s/\sqrt{n}}}{\sqrt{9/7}} \sim N(0,1)$

$$\frac{\bar{x} - \mu}{\sqrt{9s^2/7n}} \sim N(0,1)$$

$$-2.58 < \frac{\bar{x} - \mu}{\sqrt{9s^2/7n}} < 2.58$$

$$50 \pm 2.58 \sqrt{\frac{9(48.667)}{7(10)}}$$

$$CI_{99\%} = (43.54, 56.45)$$

10) A type 1 error is the probability of rejecting H_0 given that it is true.

$X \sim Poi(\lambda)$

$\Sigma X \sim Poi(n\lambda)$ $n = 1000$

$$\frac{\Sigma X - n\lambda}{\sqrt{n\lambda}} \dot{\sim} N(0,1)$$

$H_0: \lambda = 3$ $H_1: \lambda \neq 3$

i) $P[\text{Type } 1] = P[\text{Rejecting } H_0 \mid H_0 \text{ is true}]$

$$= P[\Sigma X > 3100 \mid \lambda = 3]$$

$$= P\left[Z > \frac{3100 - 3000}{\sqrt{3000}}\right.$$

$= 1 - P[Z < 1.8257]$

$= P[Z > 1.8257]$

$= 0.03394$

$= 3.394\ \%$

ii) $P[\text{type } 2] = P[\text{Accepting } H_0 \mid H_0 \text{ is false}]$

$= P[\Sigma x < 3100 \mid \lambda \neq 3]$

$= P\left[Z < \dfrac{3100 - n\lambda}{\sqrt{n\lambda}}\right]$ as $\lambda \neq 3$.

$= P\left[Z < \dfrac{3100 - 1000\lambda}{\sqrt{1000\lambda}}\right]$

iii) Power of test $= 1 - P[\text{type II}]$

$= 1 - P\left[Z < \dfrac{3100 - 1000\lambda}{\sqrt{1000\lambda}} \,\middle|\, \lambda \neq 3\right]$

$= P\left[Z > \dfrac{3100 - 1000\lambda}{\sqrt{1000\lambda}} \,\middle|\, \lambda \neq 3\right]$

iv) $\hat{\lambda} = \dfrac{2900}{1000} = 2.9$

$\hat{\lambda} \pm 2.5758 \sqrt{\dfrac{\hat{\lambda}}{n}}$

$2.9 \pm 2.5758 \sqrt{\dfrac{2.9}{1000}}$

2.9 ± 0.1387

$\Rightarrow \quad CI_{99\%} = (2.7613, 3.0387)$

11) $\Sigma x = 213200$

$\Sigma x^2 = 4919860000$

$\bar{x} = 17767$

$s = 10144$

$n = 12$.

Old $\bar{x} = 17767$.

$\Sigma x = 213200 - 11000 - 48000 + 27500 + 31500$

$\therefore$ New $\Sigma x = 213200$.

New $\bar{x} = \frac{213200}{12}$.

$= 17767$.

$\therefore$ Sample mean does not change.

$\Sigma x^2 = 4919860000 - 11000^2 - 48000^2 + 27500^2 + 31500^2$

$= 4243360000$

$$s = \sqrt{\frac{\Sigma x^2 - n(\bar{x})^2}{n-1}}.$$

$$\sqrt{\frac{4243360000 - 12(17762)^2}{11}}$$

$S = 6436.033$

The sample standard deviation has reduced but the sample mean remains the same. This is because the spread between the two new values is less than the spread between the two old values.

12) $x_i \sim U(0, \theta)$

$$f(x) = \frac{1}{\theta - 0} \quad \theta > 0.$$

$$= \frac{1}{\theta}$$

$$L(\theta) = \begin{cases} 1/\theta^n, & \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

$0 \le x_1, x_2 \dots x_n \le \theta$

To maximise $\frac{1}{\theta^n}$, θ^n needs to be minimised

$\theta \geq x_{max} \quad \therefore L = x_{max}.$

$\ln L(\theta) = n \ln \theta \Rightarrow \theta$

$$\frac{d \ln(L(\theta))}{d\theta} = -\frac{n}{\theta}$$

$$\frac{d^2 \ln\theta}{d\theta} = \frac{n}{\theta^2} > 0.$$

As second order derivative of $L(\theta)$ is always positive, CRLB for the variance for the unbiased estimator of θ is not pos

$\hat{\theta}(c) = cY, \; Y = x_{max}$.

ii) $P[Y \leq y] = P[X_{max} \leq y]$
$= P[X_1, X_2 \cdots X_n \leq y]$
$= P[X_1 \leq y]\, P[X_2 \leq y] \cdots P[X_n \leq y]$
$= P[X_i \leq y]^n$

$$= \left[\int_0^y \frac{1}{\theta}\, dx\right]^n$$

$$= \left(\left[\frac{x}{\theta}\right]_0^y\right)^n = \left(\frac{y}{\theta}\right)^n = F_Y(y)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{y}{\theta}\right)^n$$
$$= \frac{n y^{n-1}}{\theta^n}, \quad 0 < y < \theta$$

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$$E(Y^k) = \int_0^\theta y^k \frac{n}{\theta^n} y^{n-1}\, dy$$

$$= \frac{n}{\theta^n} \int_0^\theta y^{n+k-1}\, dy$$

$$= \frac{n}{\theta^n} \left[\frac{y^{n+k}}{n+k} \right]_0^\theta$$

$$= \frac{n}{\theta^n (n+k)} \left[\theta^{n+k} \right]$$

$$= E\left[Y^k\right] = \frac{n\theta^k}{n+k}, \quad k > 0.$$

iii) $\text{Bias}\left[\hat{\theta}(c)\right] = E\left[\hat{\theta}(c)\right] - \theta$

$$= E\left(cY\right) - \theta$$

$$= c\left[\frac{n\theta^1}{n+1}\right] - \theta$$

$$= \frac{cn\theta}{n+1} - \theta$$

$$\text{Bias}\left[\hat{\theta}(c)\right] = \theta\left[\frac{cn}{n+1} - 1\right]$$

iv) —

v) —

vi) —