→ Probability and Statistics Assignment

**Q1**

$X \sim$ Home Insurance $\sim N(800, 10000)$

$Y \sim$ Car Insurance $\sim N(1200, 90000)$

→ $P[3 \text{ Car} > 4 \text{ Home} + 800] \cdot P[(Y_1+Y_2+Y_2) - (X_1+X_2+X_3+X_4) > 800]$

→ $P[3Y - 4X > 800]$

$\therefore Y_1+Y_2+Y_3 - (X_1+X_2+X_3+X_4) \sim N[3(1200) - 4(800),\ 3(90{,}000) + 4(10{,}000)]$

$Z \rightarrow \sim N[400,\ 310{,}000]$

$\therefore P\left[Z > \frac{800-\mu}{SD}\right] = P\left[Z > \frac{800-400}{\sqrt{310000}}\right] = P[Z > 0.718]$

$= 1 - P[Z \le 0.718] = \underline{0.236}$

**Q2** MGF of $Y = M_Y(t) = M_N[\log M_X(t)]$

$= M_N\left[\log\left(1 - \frac{t}{\lambda}\right)^{-\alpha}\right]$ , $N \sim$ Neg Binomial

$X \sim$ Gamma $(\alpha, \lambda) = (6, 2)$

→ $M_N\left[-6 \cdot \log\left(1 - \frac{t}{2}\right)\right]$ → $p = 0.9$ $q = 0.1$

$\therefore M_Y(t) = \left[\frac{0.9e^t}{1-0.1e^t}\right]^k = \left[\frac{0.9\, e^{\ln_e(1-t/2)^{-6}}}{1 - 0.1e^{\ln(1-t/2)^{-6}}}\right]^k$

$\cancel{0.9(-6\ln(-t-} = \frac{\cancel{0.9(-6)\ln(1-t/2)}}{1-0.1(-6)[\ln(1-}\quad \left[\frac{0.9(-6)[1-t/2]}{1+0.1[1-t/2](6)}\right]^k$

$\therefore M_Y(t) = \left[\frac{-5.4(1-0.5t)}{1+0.6(1-0.5t)}\right]^k$ $p^k = 0.9^3$ $\therefore \underline{k=3}$

ii) $E(Y) = M'_Y(0)$

$V(x) = M''_Y(0) - [M'_Y(0)]^2$

Q3 i) $f(x) = \lambda e^{-\lambda(x-a)}$

$\therefore M_x(t) = E(e^{tx}) = \int_a^\infty e^{tx} \cdot f_x(x)\,dx = \int_a^\infty e^{tx} \times \lambda e^{-\lambda(x-a)}\,dx$

$= \lambda \times \int_a^\infty e^{tx-\lambda x+\lambda a}\,dx = \lambda \int_a^\infty e^{x[t-\lambda]} \times e^{\lambda a}\,dx$

$\therefore \lambda \times e^{\lambda a} \times \int_a^\infty e^{x(t-\lambda)}\,dx = \lambda \times e^{\lambda a} \times \left[\frac{e^{x(t-\lambda)}}{(t-\lambda)}\right]_a^\infty$

$= \lambda e^{\lambda a} \times \left[e^{\infty} - \frac{e^{a(t-\lambda)}}{(t-\lambda)}\right]$ $\therefore MGF = \frac{-\lambda \times e^{at-\cancel{\lambda a}+\cancel{\lambda a}}}{(t-\lambda)}$

$\therefore$ MGF of $X = \frac{-\lambda \times e^{ta}}{t-\lambda}$

ii.)

Q.4 $G_v(t) = \left(\frac{p}{1-qt}\right)^k$, $\rightarrow G_v(t) = \sum t^v \cdot P[V=v]$

$\therefore G_v(t) = p^k \times [1-(1-p)t]^{-k} = p^k \times (1-t+tp)^{-k}$

$G'_v(t)$: Derivative: $p^k[-k(1-t+tp)^{-k-1} \times (-1+p)]$

$\therefore p^k[-k(1-t+tp)^{-k-1} \times (-q)]$

$G''''_v(t) = p^k q^4 k(k+1)(k+2)(k+2)(1-qt)^{-(k+4)}$

$\therefore P[V=4] = \frac{t^4}{4!} p^k (-p)^4 \times k(k+1)(k+2)(k+3)$

Q.5 $f_{XY}(xy) = ax(x+y)$

$\rightarrow \int_0^5 \int_{10}^{20} ax\,[x+y]\,dy\,dx = \int_0^5 ax \int_{10}^{20} x+y\,dy\,dx.$

$= \int_0^5 ax\left[xy + \frac{y^2}{2}\right]_{10}^{20} dx = \int_0^5 ax\,(10x + 150)\,dx$

$= a\left[10\right.\ \ a\int_0^5 10x^2 + 150x\,dx = a\left[\frac{10x^3}{3} + \frac{150x^2}{2}\right]_0^5$

$= a\left\{\frac{10\times125}{3} + \frac{150(25)}{2}\right\} = a\left(\frac{1250}{3} + 1875\right)$ ($750$, $416.6$)

$\therefore a = \frac{1}{2291.6}$ $\boxed{a = 0.00043}$

$\rightarrow E(Y/X) = \int_y y \cdot f(x, y/x)\,dy.$

$f(y) = \int_0^5 ax^2 + axy\,dx = \left[\frac{ax^3}{3}\right]_0^5 + ay\left[\frac{x^2}{2}\right]_0^5$

$= a \times \frac{125}{3} + ay\frac{25}{2}$

$f(x) = \int_{10}^{20} ax(x+y)\,dy = ax\left[xy + \frac{y^2}{2}\right]_{10}^{20}$

$= 10ax^2 + 150ax$

$E(Y/X) = \int_{10}^{20} y \times \frac{ax(x+y)}{10ax(x+150)}\,dy = \frac{1}{10(x+150)}\int_{10}^{20} xy + y^2\,dy.$

$= \frac{1}{10(x+150)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right] = \frac{1}{10(x+150)}\left[150x + \frac{7000}{3}\right]$

$\rightarrow P[Y > 1/3\,x + 10] = \int_0^5 \int_{10}^{(1/3\,x+10)} f_{XY}(xy)\,dy\,dx$

$= \int_0^5 \left[ax^2y + \frac{axy^2}{2}\right]_{10}^{10+x/3} dx = \int_0^5 ax^2 \times \frac{x}{3} + \frac{ax}{2}\left[\left(\frac{10+x}{3}\right)^2 - 100\right]$

$$\int_0^5 \frac{ax^3}{3} + \frac{ax}{2} \left\{ \cancel{100} + \frac{x^2}{9} - \frac{20x}{3} - \cancel{100} \right\} dx$$

$$\to \int_0^5 \frac{ax^3}{3} + \frac{ax}{2} \cdot \frac{ax^3}{18} - \frac{20ax^2}{6\,3} dx$$

$$= \frac{125a}{3} + \frac{125a}{18} - \frac{250a}{9}$$

Q.6 $X \sim Poi(\lambda)$ $Y \sim Poi(\mu)$ $X - Y = Z \sim Poi(\lambda - \mu)$

$\to$ MGF of $Z = E(e^{tz}) = E(e^{tx - ty}) = E\left(\frac{e^{tx}}{e^{ty}}\right) = \frac{M_x(t)}{M_y(t)}$

MGF $\to$ $\frac{e^{\lambda(e^t-1)}}{} : \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}} = e^{\lambda(e^t-1) + [-\mu(e^t-1)]}$

$= e^{\lambda(e^t-1)} \times e^{-\mu(e^t-1)} = e^{(e^t-1)[\lambda-\mu]}$

Q8

ii) MGF of $Z$: $E(e^{tz}) = E(e^{tx+ty}) = E(e^{tx}) \cdot E(e^{ty})$

$= M_x(t) \cdot M_y(t) = e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)}$

$\therefore$ Thus $Z \sim Poi(\lambda+\mu)$

Q.7 iii) $V[X/Y=2] = E(x^2/y) - [E(x/y)]^2$

$= E(x^2/y=2) - [E(x/y=2)]^2$

$= \sum x^2 \cdot \frac{P(x=x, y=2)}{P(y=2)} - \left[\sum x \cdot \frac{P(x=x, y=2)}{P(y=2)}\right]^2$

Q.9

$$= \left\{\left(1^2 \cdot \frac{0}{0.6}\right) + \left(2^2 \cdot \frac{0.3}{0.6}\right) + \left(3^2 \cdot \frac{0.1}{0.6}\right) + \left(4^2 \cdot \frac{0.2}{0.6}\right)\right\} -$$

$$\left[\left(\frac{1 \times 0}{0.6}\right) + \left(\frac{2 \times 0.3}{0.6}\right) + \left(\frac{3 \times 0.1}{0.6}\right) + \left(\frac{4 \times 0.2}{0.6}\right)\right]^2$$

$$= \left(0 + 2 + 15 + \frac{16}{3}\right) - \left[0 + 1 + 5 + \frac{4}{3}\right]^2$$

$$= \left(\frac{67}{3}\right) - \left(\frac{22}{3}\right)^2 = \frac{4005}{9}$$

ii) $E(u/v=v) = \int_0^1 u \times f(u/v=v) = \int_0^1 u \cdot \frac{f(uv)}{f_v} du$

$f_v(v) = \int_0^1 \frac{48}{67}(2uv - u^2) = \frac{48}{67} \times \frac{1}{3}(3v-1)$

$\therefore E(u/v=v) = \frac{\int_0^1 u \times \frac{48}{67}(2uv-u^2)\, du}{\frac{48}{67 \times 3}(3v-1)}$

$\left(\frac{1}{3v-1}\right) \times \left\{ v \cdot 2\int_0^1 u^2 - \int_0^1 u^2 \right\} du$

$= \frac{2v(1) - 1}{3(3v-1)} = \frac{2v-1}{9v-3}$

Q8 $X \sim Gamma(3,2)$ $E(Y/X=x) = 3x+1$

$V(Y/X=x) = 2x^2+5$

$E(Y) = E(E(Y/X=x)) = E(3X+1) = 3E(x)+1 =$

$E(x) = \alpha/\lambda = 3/2 = 1.5 \therefore E(Y) = 3(1.5)+1 = 5.5$

$\rightarrow V(Y) = E[V(Y/X=x)] + V[E(Y/X=x)]$

$= E(2x^2+5) + V(3X+1)$

$= 2E(x^2) + 5 + 9V(x)$

$= 2\cdot[V(x) + E(x)^2] + 9V(x) + 5$

$V(x) = \alpha/\lambda^2 = 3/4 \rightarrow 2\left(3/4 + 9/4\right) + 9 \times 3/4 + 5$

$= 6 + 6.75 + 5 = \underline{17.75}$

Q.9 $E(X) = 3$ $V(X) = 4$ $E(Y) = 4$ $V(Y) = 1$

$Corr(X,Y) = -0.3$ $Z = X+Y$

i) $Cov(XZ) = E(XZ) - E(x)E(Z)$

$Z = X+Y$

$E(Z) = E(X) + E(Y) = 7$

$E(XZ) = E(X(X+Y)) = E(X^2+XY) = E(X^2) + E(XY)$

$\rightarrow \therefore E(X^2) = 4 + 9 = 13$

$\rightarrow$

$Cov(XY) = E(XY) - E(x)E(y)$

$(-0.3\times 2) = E(XY) - \cancel{3\times 4}(3\times 4)$.

$\therefore E(XY) = 12 - 0.6 = 11.40$

$\rightarrow E(XZ) = E(X^2) + E(XY) = 13 + 11.40 = \underline{\underline{24.40}}$

$\rightarrow Cov(XZ) = 24.40 - 3(7) = 3.40$

$$\boxed{Corr(XY) = \frac{E(xy) - E(x)(Ey)}{SDx\ SDy}}$$

ii) $V(z) = Cov[x+y, x+y] = V(x) + V(y) + (Cov(xy)\times 2)$

$= \cancel{4+1+2(11.40)}$

$= 4+1+2(-0.3)$

$= 5-(0.6\times 2) = \boxed{3.8 = Var(z)}$

Q.10 $f_x(x,y) = kx^{-a}.e^{-y/\beta}$

$\therefore \int_1^\infty \int_1^\infty kx^{-a}.e^{-y/\beta}\,dy\,dx = 1$

$\rightarrow \int_1^\infty kx^{-a} \times \left[\frac{e^{-y/\beta}}{-1/\beta}\right]_1^\infty dx = \int_1^\infty kx^{-a}.\left[0 + \frac{e^{-1/\beta}}{-1/\beta}\right]dx$

$= \int_1^\infty kx^{-a}.\,e^{-1/\beta} \times \beta\,dx = ke^{-1/\beta} \times \beta \times \int_1^\infty x^{-a}\,dx$.

$= ke^{-1/\beta} \times \beta \times \left[\frac{x^{-a+1}}{-a+1}\right]_1^\infty = ke^{-1/\beta} \times \beta\left[0 - \frac{1}{(-a+1)}\right]$

$= ke^{-1/\beta} \times \beta\left(\frac{1}{1-a}\right) = 1$

$\therefore k = \frac{1-a}{\beta} \times \frac{1}{e^{-1/\beta}} \quad \therefore \boxed{k = \frac{(1-a)}{\beta} e^{1/\beta}}$