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Section - A
Roll NO - 42

CALCULUS Assignment - 1

Q.1] Evaluate the following limit

$$\lim_{h\to 0} \frac{2(-3+h)^2-18}{h} = \lim_{h\to 0} \frac{2(9-6h+h^2)-18}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{18}-12h+2h^2-\cancel{18}}{h}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}(-6+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (-12+2h)$$

$$= -12+2(0)$$

$$= -12$$

$$\boxed{\therefore \lim_{h\to 0} \frac{2(-3+h)^2-18}{h} = -12}$$

Q.2] Determine if the following function is continuous or discontinuous at a) $x=4$ b) $x=6$.

$$g(x) = \begin{cases} 2x & x<6 \\ x-1 & x \geq 6. \end{cases}$$

a) $x=4$

Put $x=4$ in $g(x)$ $\therefore g(4) = 2(4) = 8$

$\therefore g(4)$ is defined

$$\lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} 2x = 2(4) = 8$$

$$\lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} 2x = 2(4) = 8$$

$$\therefore \lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x)$$

Hence $\lim_{x \to 4} g(x)$ exists.

$$\therefore \lim_{x \to 4} g(x) = 8$$

$$g(4) = \lim_{x \to 4} g(x) = 8$$

$\boxed{\therefore \text{The given function, } g(x) \text{ is continuous at } x = 4.}$

b) $x = 6$

put $x=6$ in $g(x)$

$\therefore g(6) = 6-1 = 5$ $\quad \therefore g(6) = 5$

$\therefore g(6)$ is defined

$$\lim_{x \to 6^-} g(x) = \lim_{x \to 6^-} 2x = 2(6) = 12$$

$$\lim_{x \to 6^+} g(x) = \lim_{x \to 6^+} x - 1 = 6 - 1 = 5$$

$\therefore \lim_{x \to 6^-} g(x) \neq \lim_{x \to 6^+} g(x)$

$\therefore \lim_{x \to 6} g(x)$ does not exist.

$\boxed{\therefore \text{The given function } g(x) \text{ is discontinuous at } x = 6}$

Q.3] Solve the following limit.

$$\lim_{x \to 9} \frac{x-9}{3-\sqrt{x}} = \lim_{x \to 9} \frac{x-9}{3-\sqrt{x}} \times \frac{3+\sqrt{x}}{3+\sqrt{x}}$$ — Rationalizing the denominator

$$= \lim_{x \to 9} \frac{(x-9)(3+\sqrt{x})}{(3)^2-(\sqrt{x})^2}$$ $- [(a-b)(a+b) = a^2 - b^2]$

$$= \lim_{x \to 9} \frac{(x-9)(3+\sqrt{x})}{(9-x)}$$

$$= \lim_{x \to 9} \frac{\cancel{(x-9)}(3+\sqrt{x})}{-\cancel{(x-9)}}$$

$$= \lim_{x \to 9} -(3+\sqrt{x})$$

$$= -(3+\sqrt{9})$$

$$= -(3+3)$$

$$= -6$$

$$\boxed{\therefore \lim_{x \to 9} \frac{x-9}{3-\sqrt{x}} = -6}$$

Q.4] Determine if the following function is continuous or discontinuous at

a) $x = -1$ b) $x = 0$ c) $x = 3$.

$$f(x) = \frac{4x+5}{9-3x}$$

a) $x = -1$

Put $x=-1$ in $f(x)$

$$\therefore f(-1) = \frac{4(-1)+5}{9-3(-1)} = \frac{-4+5}{9+3} = \frac{1}{12}$$

$\therefore f(-1)$ is defined

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{4x+5}{9-3x} = \frac{4(-1)+5}{9-3(-1)} = \frac{-4+5}{9+3} = \frac{1}{12}$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{4x+5}{9-3x} = \frac{4(-1)+5}{9+3} = \frac{1}{12}$$

$$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = \frac{1}{12}$$

$\therefore \lim_{x \to -1} f(x)$ exists.

$$\lim_{x \to -1} f(x) = \frac{1}{12}$$

$$f(-1) = \lim_{x \to -1} f(x) = \frac{1}{12}$$

$\boxed{\therefore \text{The given function is continuous at } x = -1}$

b) $x = 0$

put $x=0$ in $f(x)$ $\therefore f(0) = \frac{4(0)+5}{9-3(0)} = \frac{5}{9}$

$\therefore f(0)$ is defined

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{4x+5}{9-3x} = \frac{4(0)+5}{9-3(0)} = \frac{5}{9}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{4x+5}{9-3x} = \frac{4(0)+5}{9-3(0)} = \frac{5}{9}$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \frac{5}{9}$$

$\therefore \lim_{x \to 0} f(x)$ exists.

$$\therefore \lim_{x \to 0} f(x) = \frac{5}{9}$$

$$f(0) = \lim_{x \to 0} f(x) = \frac{5}{9}$$

$\boxed{\therefore \text{The given function is continuous at } x = 0}$

c) $x = 3$

put $x = 3$ in $f(x)$

$\therefore f(3) = \frac{4(3)+5}{9-3(3)} = \frac{12+5}{9-9} = \frac{17}{0}$, which is not defined.

$\therefore f(3)$ is not defined

$\boxed{\therefore \text{The given function } \cancel{f(3)}\ f(x) \text{ is discontinuous at } x = 3}$

Q.5] Evaluate the following limit

$$\lim_{x \to \infty} \frac{6e^{4x} - 2e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}}$$

$$= \lim_{x \to \infty} \frac{\cancel{e^{4x}}(6 - 2e^{-6x})}{\cancel{e^{4x}}(8 - e^{-2x} + 3e^{-5x})}$$

$$= \lim_{x \to \infty} \left[\frac{6 - \frac{1}{e^{6x}}}{8 - \frac{1}{e^{2x}} + \frac{3}{e^{5x}}} \right]$$

$$= \frac{6 - \frac{1}{e^{6\times\infty}}}{8 - \frac{1}{e^{2\times\infty}} + \frac{3}{e^{5\times\infty}}}$$

$$= \frac{6 - e^{-\infty}}{8 - e^{-\infty} + 3 \cdot e^{\infty}}$$

$$= \frac{6 - 0}{8 - 0 + 0}$$

$$= \frac{6}{8} = \frac{3}{4}$$

$$\boxed{\therefore \lim_{x \to \infty} \frac{6e^{4x} - 2e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}} = \frac{3}{4}}$$

Q.6.] Given the function

$$g(y) = \begin{cases} y^2+5 & \text{if } y < -2 \\ 1-3y & \text{if } y \geq -2 \end{cases}$$

a) $\lim_{x \to 6} g(y) = \lim_{y \to 6} 1-3y$

$= 1-3(6)$

$= 1-18$

$= -17$

$\boxed{\therefore \lim_{x \to 6} g(y) = -17}$

b) $\lim_{y \to -2} g(y) = \lim_{x \to -2} 1-3y$

$= 1-3(-2)$

$= 1+6$

$= 7$

$\boxed{\therefore \lim_{x \to -2} g(y) = 7}$

Q.7.] Use the Product rule to find the derivative of,

$f(t) = (4t^2-t)(t^3-8t^2+12)$

Differentiating w.r.t t on both sides

$\therefore f'(t) = \cancel{(4t^2-t)(3t^2-16t)} + (4t^2-t) \cdot \frac{d}{dt}(t^3-8t^2+12) + (t^3-8t^2+12) \cdot \frac{d}{dt}(4t^2-t)$

— [By u.v rule]

$$\therefore f'(t) = (4t^2 - t)(3t^2 - 16t) + (t^3 - 8t^2 + 12)(8t - 1)$$
$$= 12t^4 - 64t^3 - 3t^3 + 16t^2 + 8t^4 - t^3 - 64t^3 + 8t^2 + 96t - 12$$
$$= 20t^4 - 132t^3 + 24t^2 + 96t - 12$$

$$\boxed{\therefore f'(t) = 20t^4 - 132t^3 + 24t^2 + 96t - 12}$$

Q.8] Use the Quotient Rule to find the derivative of :

$$R(w) = \frac{3\omega + \omega^4}{2\omega^2 + 1}$$

Differentiate w.r.t ω on both sides.

$$\therefore R'(w) = \frac{(2\omega^2+1) \cdot \frac{d}{dw}(3\omega + \omega^4) - (3\omega + \omega^4) \cdot \frac{d}{dw}(2\omega^2 + 1)}{(2\omega^2+1)^2}$$

— [By u/v rule]

$$= \frac{(2\omega^2+1)(3+4\omega^3) - (3\omega + \omega^4)(4\omega)}{(2\omega^2+1)^2}$$

$$= \frac{6w^2 + 8w^5 + 3 + 4\omega^3 - 12\omega^2 - 4\omega^5}{(2\omega^2+1)^2}$$

$$= \frac{4\omega^5 + 4\omega^3 - 6\omega^2 + 3}{(2\omega^2+1)^2}$$

$$\boxed{\therefore R'(w) = \frac{4\omega^5 + 4\omega^3 - 6\omega^2 + 3}{(2\omega^2+1)^2}}$$

Q.9] Differentiate

$f(x) = 2e^{x} - 8^{x}$

Differentiating w.r.t x on both sides.

$\therefore f'(x) = 2e^{x} - 8^{x}.\log 8$ — $\left[f'(e^{x}) = e^{x} \; , \; f'(a^{x}) = a^{x}.\log a \right]$

Q.10] Differentiate

$y = z^{5} - e^{z}.\ln(z)$

Differentiate w.r.t z on both sides,

$\therefore \frac{dy}{dz} = \frac{dz^{5}}{dz} - e^{z}\frac{d}{dz}\ln(z) + \ln(z)\cdot\frac{d}{dz}(e^{z})$

$= 5z^{4} - e^{z}.\frac{1}{z} + \ln(z)\cdot e^{z}$

$= 5z^{4} - e^{z}\left(\frac{1}{z} \mp \ln(z)\right)$

$\boxed{\therefore \frac{dy}{dz} = 5z^{4} - e^{z}\left(\frac{1}{z} - \ln(z)\right)}$

Q.11] Using chain rule differentiate the following:-

a) $f(x) = (6x^{2} + 7x)^{4}$

Differentiate w.r.t x on both sides.

$\therefore f'(x) = 4(6x^{2}+7x)^{4-3} \frac{d}{dx}(6x^{2}+7x)$

Q.9] Differentiate

$f(x) = 2e^x - 8^x$

Differentiating w.r.t x on both sides

$\therefore f'(x) = 2e^x - 8^x \cdot \log 8$ — $\left[f'(e^x) = e^x \quad , f'(a^x) = a^x . \log a \right]$

Q.10] Differentiate

$y = z^5 - e^z \ln(z)$

Differentiate w.r.t z on both sides,

$\therefore \frac{dy}{dz} = \frac{d z^5}{dz} - e^z \frac{d}{dz} \ln(z) + \ln(z) \cdot \frac{d}{dz}(e^z)$

$= 5z^4 - e^z \cdot \frac{1}{z} + \ln(z) \cdot e^z$

$= 5z^4 - e^z \left(\frac{1}{z} \cancel{+} \ln(z) \right)$

$$\boxed{\therefore \frac{dy}{dz} = 5z^4 - e^z \left(\frac{1}{z} - \ln(z) \right)}$$

Q.11] Using chain rule differentiate the following:-

a) $f(x) = (6x^2 + 7x)^4$

Differentiate w.r.t x on both sides.

$\therefore f'(x) = 4(6x^2 + 7x)^{4-3} \frac{d}{dx}(6x^2 + 7x)$

$$\therefore f'(x) = 4(6x^2+7x)^3 \cdot (12x+7)$$

b) $f(t) = 5 + e^{4t+t^7}$

Differentiate w.r.t t on both sides,

$$\therefore f'(t) = \frac{d}{dt}(5) + \frac{d}{dt} e^{(4t+t^7)}$$

$$= 0 + e^{(4t+t^7)} \cdot \frac{d}{dt}(4t+t^7)$$

$$= e^{(4t+t^7)} (4+7t^6)$$

$$\boxed{\therefore f'(t) = e^{(4t+t^7)} \cdot (4+7t^6)}$$

Q.12] Determine the second derivative of

a) $z = \ln(7-x^3)$

Differentiate w.r.t z on both sides,

$$\therefore \frac{dz}{dx} = \frac{1}{(7-x^3)} \cdot \frac{d}{dx}(7-x^3)$$

$$\therefore \frac{dz}{dx} = \frac{1}{7-x^3} \cdot (-3x^2)$$

$$\therefore \frac{dz}{dx} = \frac{-3x^2}{7-x^3}$$

Differentiate w.r.t x on both sides.

$$\therefore \frac{d^2z}{dx^2} = \frac{(7-x^3)\cdot\frac{d}{dx}(-3x^2) - (-3x^2)\cdot\frac{d}{dx}(7-x^3)}{(7-x^3)^2}$$

$$= \frac{(7-x^3)(-6x) - (-3x^2)(-3x^2)}{(7-x^3)^2}$$

$$= \frac{-42x + 6x^4 - 9x^4}{(7-x^3)^2}$$

$$= \frac{-3x^4 - 42x}{(7-x^3)^2}$$

$$\boxed{\therefore \frac{d^2z}{dx^2} = \frac{-3x^4 - 42x}{(7-x^3)^2}}$$

b) $g(v) = \frac{2}{(6+2v-v^2)^4} = 2(6+2v-v^2)^{-4}$

Differentiate w.r.t v on both sides.

$$\therefore g'(v) = 2\cdot -4(6+2v-v^2)^{-4-1}\cdot\frac{d}{dv}(6+2v-v^2)$$

$$= -8(6+2v-v^2)^{-5}\cdot(2-2v)$$

Differentiate w.r.t v on both sides,

$$g''(v) = -8\left[(6+2v-v^2)^{-5}\cdot\frac{d}{dv}(2-2v) + (2-2v)\cdot\frac{d}{dv}(6+2v-v^2)^{-5}\right]$$

$$= -8\left[(6+2v-v^2)^{-5}(-2) + (2-2v) - 5\cdot(6+2v-v^2)^{-6}(2-2v)\right]$$

$$\boxed{g''(v) = 16(6+2v-v^2)^{-5} + 40(6+2v-v^2)^{-6}(2-2v)^2}$$

c) $f(t) = \ln(1+t^2)$

Differentiate w.r.t t on both sides,

$$f'(t) = \frac{1}{(1+t^2)} \cdot \frac{d}{dt}(1+t^2) \quad - [\text{By using chain rule}]$$

$$f'(t) = \frac{2t}{1+t^2}$$

Differentiating again w.r.t t on both sides.

$$\therefore f''(t) = \frac{(1+t^2)\cdot\frac{d}{dt}(2t) - (2t)\cdot\frac{d}{dt}(1+t^2)}{(1+t^2)^2} \quad - [\text{By } u/v \text{ rule}]$$

$$= \frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2}$$

$$= \frac{2+2t^2-4t^2}{(1+t^2)^2}$$

$$= \frac{-2t^2+2}{(1+t^2)^2}$$

$$\boxed{\therefore f''(t) = \frac{-2t^2+2}{(1+t^2)^2}}$$

Q.13] Find the maximum and minimum value of the function.

$f(x) = x^3 - 3x^2 - 9x + 12$

Differentiate w.r.t x on both sides

$\therefore f'(x) = 3x^2 - 6x - 9$

Put $f'(x) = 0$

$\therefore 3x^2 - 6x - 9 = 0$

$3(x^2 - 2x - 3) = 0$

$\therefore x^2 - 3x + x - 3 = 0$

$\therefore x(x-3) + 1(x-3) = 0$

$\therefore (x-3)(x+1) = 0$

$\therefore (x-3) = 0 \quad or \quad (x+1) = 0$

$\therefore x = 3 \quad or \quad x = -1$

Differentiate $f'(x)$ w.r.t x on both sides

$\therefore f''(x) = 6x - 6$

Put $x = 3$ in $f''(x)$	Put $x = -1$ in $f''(x)$
$\therefore f''(3) = 6(3) - 6$	$\therefore f''(-1) = 6(-1) - 6$
$= 18 - 6$	$= -6 - 6$
$= 12$	$= -12$
Hence, $f(x)$ is minimum at $x = 3$ $[\because f''(x) > 0]$	$\therefore f(x)$ is maximum at $x = -1$ $[\because f''(x) < 0]$

Put $x = 3$ in $f(x)$

$\therefore$ Minimum value $= f(3) = (3)^3 - 3(3)^2 - 9(3) + 12$

$= 27 - 3 \times 9 - 9 \times 3 + 12$

$= \cancel{27} - \cancel{27} - 27 + 12$

$= -15$

Put $x=-1$ in $f(x)$

$\therefore$ Maximum value $= f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 12$

$= (-1) - 3 \times 1 + 9 - 12$

$= -1 - 3 + 9 + 12$

$= \cancel{9-16} \quad -4 + 21$

$= \cancel{-7} \quad 17$

$\therefore$ The given function ~~is~~ $f(x)$ has the maximum value 17 at $x=-1$ & has the minimum value -15 at $x=3$

Q.14] Find the minimum and maximum value of the given function.

$f(x) = 4x^3 - 18x^2 + 24x - 7$

Differentiate w.r.t x on both sides

$\therefore f'(x) = 12x^2 - 36x + 24$

Put $f'(x) = 0$

$\therefore 12x^2 - 36x + 24 = 0$

$\therefore 12(x^2 - 3x + 2) = 0$

$\therefore x^2 - 2x - x + 2 = 0$

$\therefore x(x-2) - 1(x-2) = 0$

$\therefore (x-2)(x-1) = 0$

$\therefore (x-1) = 0$ or $(x-2) = 0$

$\therefore x = 1$ or $x = 2$

Differentiate $f'(x)$ w.r.t x on both sides

$\therefore f''(x) = 24x - 36$

put $x = 1$ in $f''(x)$

$\therefore f''(1) = 24(1) - 36 = 24 - 36 = -12$ i.e $f''(x) < 0$

$\therefore f(x)$ is maximum at $x = 1$

Put $x=2$ in $f''(x)$

$\therefore f''(2) = 24(2) - 36 = 48 - 36 = 12$ i.e $f''(x) > 0$

$\therefore f(x)$ is minimum at $x = 2$

Put $x = 2$ in $f(x)$

$\therefore$ Minimum value $= f(2) = 4(2)^3 - 18(2)^2 + 24(2) - 7$

$= 4 \times 8 - 18 \times 4 + 24 \times 2 - 7$

$= 32 - 72 + 48 - 7$

$= 80 - 79$

$= 1$

Put $x = 1$ in $f(x)$

$\therefore$ Maximum value $= f(1) = 4(1)^3 - 18(1)^2 + 24(1) - 7$

$= 4 - 18 + 24 - 7$

$= 28 - 25$

$= 3$

$\therefore$ The given function, $f(x)$ has the maximum value 3 when $x = 1$ and has the minimum value 1 when $x = 2$

Q.15] Sketch the curve for the following function:-

$f(x) = x^2(x+3)$

Differentiating w.r.t x on both sides,

$\therefore f'(x) = 2x(x+3) + (1)x^2$ — [By u.v rule]

$= 2x^2 + 6x + x^2$

$= 3x^2 + 6x$

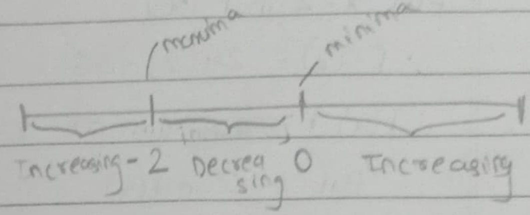
Put $f'(x) = 0$

$\therefore 3x^2 + 6x = 0$ $\quad \therefore 3(x^2 + 2x) = 0$ $\quad \therefore x^2 + 2x = 0$

$\therefore \cancel{3}x(x+2) = 0$

$\therefore \quad 3x = 0 \quad \text{or} \quad (x+2) = 0$

$\therefore x = 0 \quad \text{or} \quad x = -2$



Differentiating $f'(x)$ w.r.t x on both sides

$\cancel{\therefore f''(x) = 3x(1) + (x+3)(3)}$

$\cancel{= 3x}$

$\therefore \quad f''(x) = 6x^2 + 6$ //

Put $x = 0$ in $f''(x)$ $\therefore f''(0) = 6(0)+6 = 6$ $\quad f''(0) > 0$

Put $x = -2$ in $f''(x)$ $\therefore f''(-2) = 6(-2)+6 = -12+6 = -6$ $\quad f''(-2) < 0$

$\therefore$ The given function is maximum at $x = -2$

and minimum at $x = 0$ //

Put $x = -1$ in $f'(x)$

$\therefore f'(-1) = 3(-1)^2 + 6(-1)$

$= 3 - 6$

$= -3$ which is < 0

$\therefore$ The given function is ~~in~~decreasing between -2 and 0

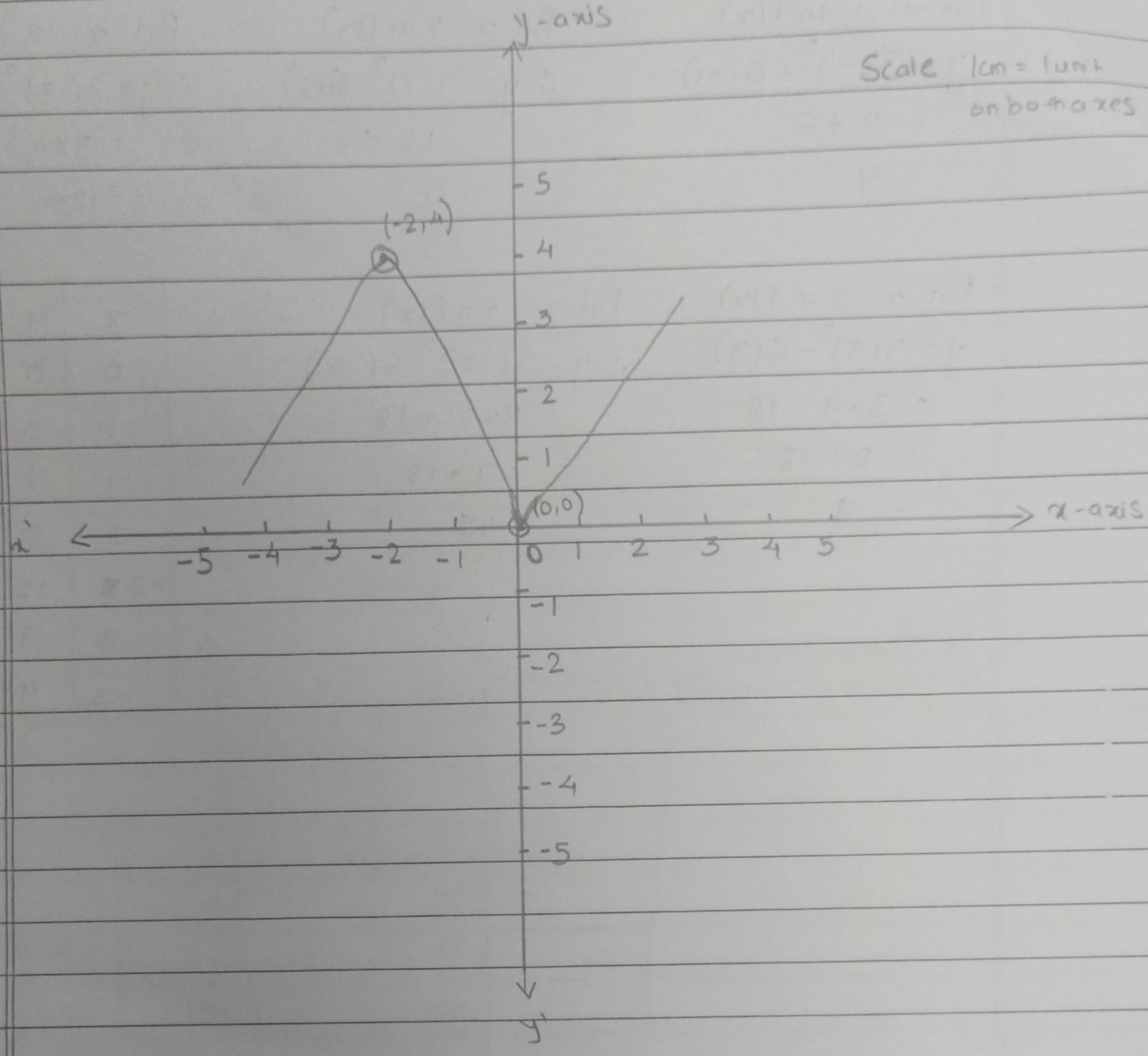
Put $x = 0$ in $f(x)$

Minimum value $= f(0) = 0(0+3) = 0$ $\quad \therefore x = 0, y = 0$

Put $x = -2$ in $f(x)$

$\therefore$ Maximum value $= f(-2) = (-2)^2(-2+3) = 4 \times 1 = 4$

$\therefore x = -2, y = 4.$



Q.16] Sketch ~~the~~ a detailed graph for the following function.

$f(x) = 3x^2 - 6x$

i.e $y = 3x^2 - 6x$

Put $x = 0$ in $f(x)$	Put $x = 1$ in $f(x)$
$\therefore y = 3(0)^2 - 6(0) = 0$	$\therefore y = 3(1)^2 - 6(1) = 3 - 6 = -3$

Put $x=-1$ in $f(x)$

$y=3(-1)^2-6(-1)$

$=3+6$

$=9$

Put $x=2$ in $f(x)$

$\therefore y=3(2)^2-6(2)$

$=12-12$

$=0$

Put $x=-2$ in $f(x)$

$\therefore y=3(-2)^2-6(-2)$

$=3\times4+12$

$=12$

Put $x=3$ in $f(x)$

$y=3(3)^2-6(3)$

$=3\times9-18$

$=27-18$

$=9$

Put $x=-3$ in $f(x)$

$\therefore y=3(-3)^2-6(-3)$

$=3\times9+18$

$=27+18$

$=45$

$\therefore$

x	y
0	0
1	-3
-1	9
2	0
-2	12
3	9
-3	45.

Scale
1cm = 5 unit on y-axis
1cm = 1 unit on x-axis