INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Numerical Methods and Algebra

Chapter: Unit 1 & 3

Category: Assignment

Assignment 1

Numerical Methods and Algebra

1. Find the value of $\sqrt{7+\sqrt{7+\sqrt{7+\cdots}}}$

2. Find the value of $\frac{x}{y}$ in the following equation:

$$7x^2 - 6y^2 = -11x^2 - 4y^2$$

3. What ordered pairs satisfy the system

$$x + 3y = 4$$
$$x^2 + y^2 - 4y = 12$$

4. The quadratic function defined by the equation $d = 2r^2 - 16r + 34$ gives the density of smoke, d, in millions of particles per cubic foot for a certain type of diesel engine. The input variable, r, represents the speed of the engine in hundreds of revolutions per minute
   a. Determine the density of smoke when r = 3.5 (350 revolutions per minute).
   b. Determine the number of revolutions per minute for minimum smoke. What is the minimum output?
   c. If the density of smoke is determined to be 100 million particles per cubic foot, determine the speed of the engine.

5. You contact two car rental companies and obtained the following information for the 1-day cost of renting a car.
   Company 1: Rs 1500 per day plus Rs 25 per km
   Company 2: Rs 2100 per day plus Rs 22 per km
   Let n represent the total number of km driven in 1 day.
   a. Write an expression to determine the total cost, C, of renting a car for 1 day from company 1.
   b. Write an expression to determine the total cost, C, of renting a car for 1 day from company 2.
   c. Use the expressions in parts a and b to write an inequality that can be used to determine for what number of km it is less expensive to rent the car from company 2.
   d. Solve the inequality.

6. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval
   a. [0, 1]
   b. [1, 3.2]
   c. [3.2, 4]

7. The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in [−1, 0] and the other in [0, 1]. Attempt to approximate these zeros to within 10^{-4} using the Newton's method.

8. You are enrolled in an algebra course at your college. You achieved grades of 70, 86, 81, and 83 on the first four exams. The final exam counts the same as an exam given during the semester.
   a. If x represents the grade on the final exam, write an expression that represents your course average (arithmetic mean).
   b. If your average is greater than or equal to 80 and less than 90, you will earn a B in the course. Using the expression from part a for your course average, write a compound inequality that must be satisfied to earn a B.
   c. Solve the inequality.

9. The NPV for a project at a discount rate of 4% is 8.54. The NPV at a discount rate of 20% is -11.81. Using the above data points, find the IRR using linear interpolation.

10. The makers of a new food delivery app estimate that with x thousand orders their monthly revenue and cost (in lakhs of Rs) are given by the following:

$$R(X) = 32x - 0.21x^2$$
$$C(x) = 195 + 12x$$

Determine the number of orders needed for the app to remain profitable.

1. To Find : $\sqrt{7+\sqrt{7+\sqrt{7+\ldots}}}$

Let $x = \sqrt{7+\sqrt{7+\sqrt{7+\ldots}}}$ —— (i)

Squaring both sides,

$x^2 = 7 + \sqrt{7+\sqrt{7+\sqrt{7+\ldots}}}$ —— (ii)

$\therefore$ Substituting (i) in (ii)

$x^2 = 7 + x$

$x^2 - x - 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-7)}}{2(1)}$$

$x = \dfrac{1 + \sqrt{29}}{2}$ or $x = \dfrac{1 - \sqrt{29}}{2}$

$\boxed{x = 3.192582404}$ or $x = -2.192582404$

N.P.

x needs to be +(ve)

$\therefore \sqrt{7+\sqrt{7+\sqrt{7+\ldots}}} = 3.192582404$

2. $7x^2 - 6y^2 = -11x^2 - 4y^2$

$7x^2 + 11x^2 = 6y^2 - 4y^2$

$18x^2 = 2y^2$

$\frac{x^2}{y^2} = \frac{2}{18} = \frac{1}{9}$

Taking roots on both sides,

$\frac{x}{y} = \pm \frac{1}{3}$

3. $x + 3y = 4$ —— (i)

$x^2 + y^2 - 4y = 12$ —— (ii)

$x = 4 - 3y$ —— (iii)

Substituting (iii) in (ii)

$(4-3y)^2 + y^2 - 4y = 12$

$16 + 9y^2 - 24y + y^2 - 4y = 12$

$10y^2 - 28y + 4 = 0$

$5y^2 - 14y + 2 = 0$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{14 \pm \sqrt{(-14)^2 - 4(5)(2)}}{2(5)}$$

$$y = \frac{7 + \sqrt{39}}{5} \quad \underline{or} \quad y = \frac{7 - \sqrt{39}}{5}$$

$y = 2.6489996$ $\underline{or}$ $y = 0.1510004$

$\therefore$ when $y = 2.6489996$, $x = -3.9469988$ || $y = 0.1510004$, $x = 3.5469988$

$\therefore$ The ordered pairs of systems are $(-3.95, 2.65)$ and $(3.55, 0.15)$

4. $d = 2r^2 - 16r + 34$

where, d = density of smoke

r = speed of engine

a) When $r = 3.5$

$d = 2(3.5)^2 - 16(3.5) + 34$

$= 2.5$ million particles per cubic foot

b) $d = 2r^2 - 16r + 34$

$\frac{d(d)}{dr} = 4r - 16$

For minimum,

$\frac{d(d)}{dr} > 0$

$4r > 16$

$r > 4$

$\therefore$ The no. of revolutions for minimum smoke = 4 per min.

Minimum output $= 2(4)^2 - 16(4) + 34$

$= 2$

c) when $d = 100$

$2r^2 - 16r - 66 = 0$

$r^2 - 8r - 33 = 0$

$(r - 11)(r + 3) = 0$

$\boxed{r = 11}$ or $r = -3$

N.D.

$\therefore$ Speed of engine = 1100 revolutions per minute.

5. Company 1 : 1500 per day + 25 per km
Company 2 : 2100 per day + 22 per km
Let n be the total no. of km driven in a day

a) Total cost for Company 1 $= 1500 + 25n$
b) Total cost for Company 2 $= 2100 + 22n$

c) For it to be less expensive to rent for C 2 :
Cost of Company 1 > Cost of Company 2
$$1500 + 25n > 2100 + 22n$$

d) $1500 + 25n > 2100 + 22n$
$$3n > 600$$
$$n > 200$$

$\therefore$ It is less expensive to rent from Company 2 when km driven in a day is greater than 200

6. $f(x) = x^3 - 7x^2 + 14x - 6 = 0$

a) Considering $a = 0$, $b = 1$

$f(a) = f(0) = -6$

$f(b) = f(1) = 2$

Since $f(a)$ & $f(b)$ have opposite signs, this interval is appropriate.

a	b	$f(a)$	$f(b)$	$(a+b/2)$	$f(a+b/2)$
0	1	-6	2	0.5	-0.625
0.5	1	-0.625	2	0.75	0.9843
0.5	0.75	-0.625	0.984375	0.625	0.259766
0.5	0.625	-0.625	0.259766	0.5625	-0.1618652
0.5625	0.625	-0.1618652	0.259766	0.59375	0.0540466
0.5625	0.59375	-0.1618652	0.0540466	0.578125	-0.0526238
0.578125	0.59375	-0.0526238	0.0540466	0.5859375	0.0010313
0.578125	0.5859375	-0.0526238	0.0010313987	0.58203125	-0.0257160
0.58203125	0.5859375	-0.025716007	0.0010313987	0.58398437	

$\therefore x = 0.58$

b) Considering $a = 1$, $b = 3.2$

$f(a) = f(1) = 2$

$f(b) = f(3.2) = -0.112$

Since $f(a)$ and $f(b)$ have opposite signs, this interval is appropriate.

a	b	$f(a)$	$f(b)$	$a+b/2$	$f(a+b/2)$
1	3.2	2	-0.112	2.1	1.791
2.1	3.2	1.791	-0.112	2.65	0.552125
2.65	3.2	0.552125	-0.112	2.925	0.08582813
2.925	3.2	0.08582813	-0.112	3.0625	-0.05444359
2.925	3.0625	0.08582813	-0.05444336	2.99375	0.006327881
2.99375	3.0625	0.006327881	-0.05444336	3.028125	-0.02652072
2.99375	3.028125	0.006327881	-0.02652072	3.0109375	-0.010696934
2.99375	3.0109375	0.006327881	-0.010696934	3.00234375	-0.0023275
2.99375	3.00234375	0.006327881	-0.00233275	2.9991211	0.001960747
2.9991211	3.00234375	0.001960747	-0.00233275	3.00019531	-1.95236198

$x = 3.00$

c) Considering $a = 3.2$, $b = 4$

$f(a) = f(3.2) = -0.112$
$f(b) = f(4) = 2$
Since $f(a)$ and $f(b)$ have opposite signs, this interval is appropriate.

a	b	$f(a)$	$f(b)$	$a+b/2$	$f(a+b/2)$
3.2	4	-0.112	2	3.6	0.336
3.2	3.6	-0.112	0.336	3.4	-0.016
3.4	3.6	-0.016	0.336	3.5	0.125
3.4	3.5	-0.016	0.125	3.45	0.046125
3.4	3.45	-0.016	0.046125	3.425	0.01301563
3.4	3.425	-0.016	0.01301563	3.4125	-0.00199805
3.4125	3.425	-0.001999805	0.01301563	3.41875	0.005381592
3.4125	3.41875	-0.00199805	0.00538159	3.415625	0.001660065

$x = 3.42$

7. $f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$

$\therefore$ $f'(x) = 920x^3 + 54x^2 + 18x - 221$

Since $f(x)$ has a real zero in $[-1, 0]$ we consider midpoint -0.5 as x_0

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -0.5 - \frac{f(-0.5)}{f'(-0.5)}$$
$$= -0.1504524887$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.041816814$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -0.0406593435$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \boxed{-0.04065928832}$$

Since $f(x)$ has a real zero in $[0, 1]$ we consider the midpoint 0.5 as x_0

$\therefore$ $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -0.7050894204$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -0.323791142$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -0.06460313103$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -0.004068615115$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = \boxed{-0.04065928835}$$

8. Grades of four exams $= 70, 86, 81$ and 83
Grades of final exam $= x$

a) Course average $= \dfrac{70+86+81+83+x}{5} = \dfrac{320+x}{5}$

b) The grade earned is B,

$\therefore \quad 80 \leq \dfrac{320+x}{5} < 90$

c) $80 \leq \dfrac{320+x}{5} < 90$

$400 \leq 320 + x < 450$

$80 \leq x < 130$

Assumption : Maximum marks $= 100$

$\therefore \quad 80 \leq x \leq 100$

$\therefore$ The final marks should be between 80 and 100 for 'B' grade.

9. $x_1 = 8.54$

$x_2 = -11.81$

$x = 0$

$y_1 = 4\% = 0.04$

$y_2 = 20\% = 0.20$

Using linear interpolation:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y = \frac{0.16 \times (-8.54)}{20.35} + 0.04$$

$$y = 0.1071449631 4$$

$$= \boxed{10.714496\% = IRR}$$

10. Revenue $= R(x) = 32x - 0.21x^2$

Cost $= C(x) = 195 + 12x$

To remain profitable, the revenues must exceed the cost:

$\therefore R(x) > C(x)$

$32x - 0.21x^2 > 195 + 12x$

$0.21x^2 - 20x + 195 < 0$

Equating the inequality to 0:

$0.21x^2 - 20x + 195 = 0$

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(0.21)(195)}}{2(0.21)}$$

$x = 84.21142753$ or $x = 11.02666771$

$\therefore 11.026667 < x < 84.21142 \rightarrow$ satisfies the equation

ie. $12 \leq x \leq 84$

$\therefore$ No. of orders needed to be profitable is more than 12 and less than 84.