SRM2 - Assignment 2 411

Q1.

(i) Writing the state space in the order {Bid (B), Offer (O)}, the generator matrix is:

$$\begin{array}{c} B \\ O \end{array} \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

(ii) The holding times are exponentially distributed with parameter λ in state B, μ in state O.

(iii) $$\frac{\partial}{\partial t} {}_tP_s^{BB} = -\lambda \cdot {}_tP_s^{BB} + \mu \cdot {}_tP_s^{BO}$$

$$\frac{\partial}{\partial t} {}_tP_s^{BO} = \lambda \, {}_tP_s^{BB} - \mu \, {}_tP_s^{BO}$$

(iv) We have a two-state model so:-

$${}_tP_s^{BB} + {}_tP_s^{BO} = 1$$

Substituting :-

$$\frac{\partial}{\partial t} {}_tP_s^{BB} = -\lambda \cdot {}_tP_s^{BB} + \mu \left(1 - {}_tP_s^{BB}\right);$$

$$\frac{\partial}{\partial t} \left[\exp((\lambda+\mu)t) \cdot {}_tP_s^{BB} \right] = \mu \exp((\lambda+\mu)t)$$

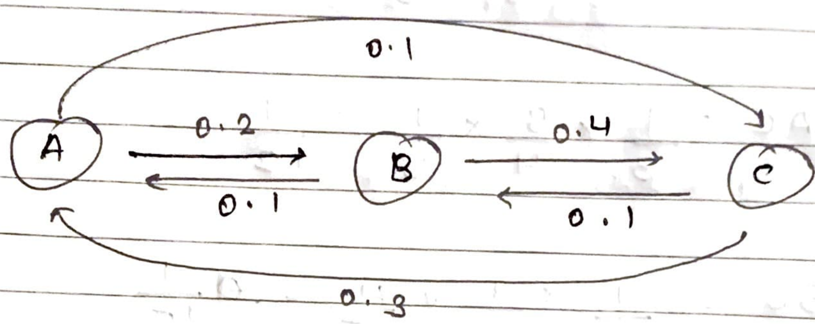
$$\exp((\lambda+\mu)t) \cdot {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} \cdot \exp((\lambda+\mu)t) + c$$

$\therefore \frac{\lambda}{\mu+\lambda}$ is constant

$${}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \exp(-(\lambda+\mu)t).$$

Q2.

i)



ii)

$$\frac{d}{dt} P_{AA}(t) = -0.3 P_{AA}(t) + 0.1 P_{AB}(t) + 0.3 P_{AC}(t)$$

$$\frac{d}{dt} P_{AB}(t) = 0.2 P_{AA}(t) - 0.5 P_{AB}(t) + 0.1 P_{AC}(t)$$

$$\frac{d}{dt} P_{AC}(t) = 0.1 P_{AA}(t) + 0.4 P_{AB}(t) - 0.4 P_{AC}(t)$$

iii) To stay in state A the equation reduces to

$$\frac{d}{dt} P_{\overline{AA}}(t) = -0.3 P_{\overline{AA}}(t)$$

$$P_{\overline{AA}}(t) = \exp(-0.3t)$$

$t = 2 \quad \exp(-0.6) = 0.5488$

(iv) The only paths under which the third jump is into state C are BAC, CAC and CBC.

The probabilities for each path are :-

$$BAC = \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} = \frac{2}{45}$$

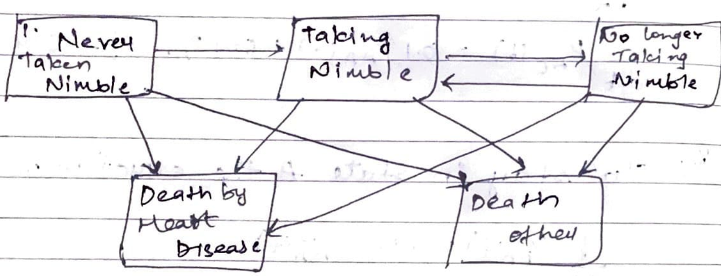
$$CAC = \frac{1}{3} \times \frac{3}{4} \times \frac{1}{8} = \frac{1}{12}$$

$$CBC = \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} = \frac{1}{15}$$

$$= 7/36$$
$$= 0.194$$

Q3.

(i)



(ii) The Chapman Kolmogorov equation

$${}_{dt+t}p_x^{34} = {}_tp_x^{31}\, {}_{dt}p_{x+t}^{14} + {}_tp_x^{32}\, {}_{dt}p_{x+t}^{24} + {}_tp_x^{33}\, {}_{dt}p_{x+t}^{34} + {}_tp_x^{34}\, {}_{dt}p_{x+t}^{44} + {}_tp_x^{35}\, {}_{dt}p_{x+t}^{54}$$

$${}_{dt+t}p_x^{34} = {}_tp_x^{32}\, {}_{dt}p_{x+t}^{24} + {}_tp_x^{33}\, {}_{dt}p_{x+t}^{34} + {}_tp_x^{34}\, {}_{dt}p_{x+t}^{44}$$

$${}_{dt}p_{x+t}^{44} = 1$$

$${}_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij}\, dt + o(dt) \qquad i \neq j$$

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$,

$${}_{dt+t}p_x^{34} = {}_tp_x^{32}\, \mu_{x+t}^{24}\, dt + {}_tp_x^{33}\, \mu_{x+t}^{34}\, dt + {}_tp_x^{34} + o(dt)$$

$${}_{dt+t}p_x^{34} - {}_tp_x^{34} = {}_tp_x^{32}\, \mu_{x+t}^{24}\, dt + {}_tp_x^{33}\, \mu_{x+t}^{34}\, dt + o(dt)$$

Q4. (i) The mean is equal to the parameter, so there are 3 calls per hour.

(ii) The process is memoryless so the fact that Fred has not had a call for 15 minutes is irreelevanat. Expected time until next call is 20 minutes.

(iii) $P_0(0.5) = \frac{e^{-1.5}(1.5)^0}{0!}$

$P_0(0.5) = e^{-1.5}$

$= 0.2231$

(iv) The expected time that Fred is on the phone is the expected number of calls times the expected length of a call.

Per hour this is 3 calls times 7 minutes = 21 minutes.

So, the probability that the phone is engaged is

$21/60 = 0.35$.

Q5. (i)

The Chapman Kolmogorov equation is,

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\, P_{HH}(t, t+dt) + P_{HD}(x,t)\, P_{DH}(t, t+dt) + P_{HS}(x,t)\, P_{SH}(t, t+dt)$$

But $P_{DH}(t, t+dt) = 0$

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\, P_{HH}(t, t+dt) + P_{HS}(x,t)\, P_{SH}(t, t+dt)$$

Assuming,

$$P_{ij}(t, t+dt) = \lambda_{ij}(t)\, dt + o(dt) \quad i \neq j$$

$$P_{ii}(t, t+dt) = 1 - \sum_{j=i} \lambda_{ij}(t)\, dt + o(dt)$$

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\,(1 - \sigma(t)\, dt - \mu(t)\, dt) + P_{HS}(x,t)\, \rho(t, ct) + o(dt)$$

$$\therefore \frac{d}{dt} P_{HH}(x,t) = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x,t)}{dt}$$

$$= P_{HH}(x,t)\,(-\sigma(t) - \mu(t)) + P_{HS}(x,t)\, \rho(t,t)$$

(ii) The equation simplifies when considering $P_{\overline{HH}}(t)$ to $\frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))\, P_{\overline{HH}}(t)$

$$\frac{1}{P_{\overline{HH}}(0,t)} \frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))$$

$$= \frac{d}{dt} \ln P_{\overline{HH}}(t).$$

Integrate both sides :-

$$\left[\ln P_{\bar{H}H}(0,t)\right]_0^t = \int_{s=0}^{t} -\left[\sigma(s)+\mu(s)\right] ds$$

as $P_{\bar{H}H}(0) = 1$

$$P_{\bar{H}H}(0,t) = \exp -\left(\int_{s=0}^{t} (\sigma(s)+\mu(s))\, ds\right)$$

Q7 (i) The maximum likelihood estimates of the transaction intensity from state i to state j is the number of transitions from state i to state j divided by the ~~tot~~ total waiting time in state i. To estimate the transition intensities exactly we therefore need the total time spent in each state.

(ii) $$\frac{\partial}{\partial t} P_{AA}(s,t) = -P_{AA}(s,t)\mu$$

$$\frac{\partial}{\partial t} P_{AT}(s,t) = P_{AA}(s,t)\mu$$

Integrated forward equations :-

$$P_{AA}(s,t) = \exp\left[-\int_{u=s}^{t} \mu\, du\right]$$

$$P_{AT}(s,t) = \int_{u=0}^{t} P_{AA}(s,u)\cdot \mu\, du.$$

(iii) Solving for P_{AT}, we have

$$\frac{\partial}{\partial t} P_{AT}(t) = P_{AA}(t)\mu = [1 - P_{AT}(t)]\mu .$$

Using integrating factor,

$$\frac{\partial}{\partial t}\left[\exp(\mu t)\, P_{AT}(t)\right] = \mu \exp(\mu t)$$

$$\exp(\mu t)\, P_{AT}(t) = \exp(\mu t) - 1$$

$$P_{AT}(t) = 1 - \exp(-\mu t)$$

$$\therefore C\left[1 - \exp(-\mu T)\right]$$

(iv)

$$\begin{tikzcd} \boxed{\text{Active policy}} \arrow[r, "\mu"] \arrow[d, "\lambda"] & \boxed{\text{Theft claim}} \\ \boxed{\text{Lapsed}} & \end{tikzcd}$$

(v) We now have $\frac{\partial}{\partial t} P_{AA}(t) = -P_{AA}(t)(\mu + \lambda)$

So, $P_{AA}(t) = \exp(-(\mu+\lambda)t)$

$$P_{AT}(t) = \left[\frac{-\mu}{(\mu+\lambda)} \exp(-(\mu+\lambda)t)\right]_0^T$$

$$= \frac{\mu}{\mu+\lambda}\left(1 - \exp(-(\mu+\lambda)T)\right)$$

$$\therefore \text{Claims} = \frac{\mu}{\mu+\lambda}\, c\left(1 - \exp[-(\mu+\lambda)T]\right)$$

Q8.

(i) $\{0, 1, 2, 3, 4, \ldots\}$

(ii)

$$\begin{tikzcd} 0 & 1 \arrow[l, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, "\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, "\mu"] \arrow[r, bend left, "\lambda"] & 4 \arrow[l, "\mu"] \arrow[r, dashed, bend left, "\lambda"] & \cdots \arrow[l, dashed, "\mu"] \end{tikzcd}$$

(iii) Generator matrix

$$\begin{array}{c|c} \text{Lives} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots \end{matrix} \\ & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) & \end{bmatrix} \end{array}$$

etc

(iv) If a Markov jump process X_t is examined only at the times of transition, the resulting process is called the jump chain associated with X_t.

(v)

$$\begin{array}{c|c} \text{Lives} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots \end{matrix} \\ & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 \\ 0 & \mu/(\mu+\lambda) & 0 & \lambda(\mu+\lambda) & 0 \\ 0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda(\mu+\lambda) \\ 0 & 0 & 0 & \mu(\mu+\lambda) & 0 \end{bmatrix} \end{array}$$

etc.

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vi) $\left(\frac{\mu}{\mu+\lambda}\right)^3$

Q10.

(i)

$$\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -2t \times P_{\bar{A}\bar{A}}(t)$$

$$\frac{d}{dt}\left[\ln P_{\bar{A}\bar{A}}(t)\right] = -2t$$

$$\ln P_{\bar{A}\bar{A}}(s) = -s^2 + c$$

$$\therefore P_{\bar{A}\bar{A}}(s) = \exp^{-s^2}$$

(ii)

P(in first visit to B at time T in state A at t = 0)

$$= \int_0^T P(\text{remains in A to time } s) \times P(\text{transition to B in time } s, s+ds) \times P(\text{remains in B to time } T)\, ds$$

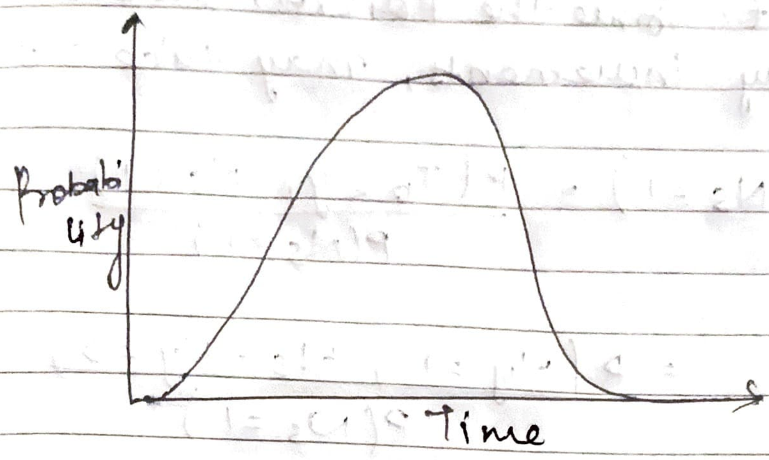
$$= \int_{s=0}^{T} P_{\bar{A}\bar{A}}(s) \times 2s \times P_{\bar{B}\bar{B}}(s, T)\, ds$$

$$= \int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2}\, ds$$

$$= \int_{s=0}^{T} 2s \times e^{-T^2}\, ds$$

$$= (e^{-T^2} \times T^2)$$

(iii) (a) The sketch should be shaped like :-



(b) - Initially probability increases from 0 at $T=0$, and accelerates as the transition rate from A to B increases.

- As transitions increase, it becomes more likely that the process has already visited state B and jumped back to A.

∴ Probability of being in the first visit to B tends (exponentially) to zero.

(c) Differentiating.

$$\frac{d}{dt}\left[e^{-t^2} \times t^2\right] = 2t\, e^{-t^2} - 2t^3 \times e^{-t^2}$$

$$e^{-t^2}\, 2t\,(1-t^2) = 0$$

$t = 1$ for a positive solution

∴ It is maxima.

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Q 11.

(i) Let N_t denote the number of claims up to time t. Since the Poisson process has stationary increments, may take = 0

$$P(T_0 \leq y \mid N_s = 1) = \frac{P(T_0 \leq y,\ N_s = 1)}{P(N_s = 1)}$$

$$= \frac{P(N_y = 1,\ N_s - N_y = 0)}{P(N_s = 1)}$$

But $N_s - N_y$ is independent of N_y
Distribution as N_{s-y}

Thus, the right hand side

$$\frac{(\lambda y e^{-\lambda y})\, e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$$

which is the cdf of uniform distribution on $[0, s]$

(ii) Holding times are independent, each having an exponential distribution, their joint density is :-

$$\lambda^n e^{-\lambda(t_1 + t_2 + \dots + t_n)},\ \{t_1, t_2, \dots t_n \geq 0$$

(iii) We have, as in part (i)

$$P(N_s = k \mid N_t\ n) = \frac{P(N_s = k,\ N_t = n)}{P(N_t = n)}$$

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$$= \frac{P(N_s = k,\ N_t - N_s = n-k)}{P(N_t = n)}$$

Using again that the Poisson process has stationary and independent increaments, that the number of claims in an interval $[0, t]$ is Poisson (λt), we derive from above that,

$$P(N_s = k \mid N_t = n) = \frac{\dfrac{e^{-\lambda s}(\lambda s)^k}{k!} \cdot \dfrac{e^{-\lambda(t-s)} \lambda^{n-k} (t-s)^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{e^{-\lambda t} \lambda^n s^k (t-s)^{n-k}}{k!\,(n-k)!} \times \frac{n!}{e^{-\lambda t} \lambda^n t^n}$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \frac{s^k (t-s)^{n-k}}{t^k\, t^{n-k}}$$

$$= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

which is binomial with parameters n and s/t.