FINANCIAL ENG

1)

(a) Arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as follows:

An Arbitrage opportunity means that :

i. We can start at time 0 with a portfolio that has a net value of zero (implying that we are in some assets and short in others.) This is usually called a zero-cost portfolio.

ii. at some future time T :

- The probability of a loss is 0.
- The probability that we make a strictly positive profit is greater than 0.

If such an opportunity existed then we could multiply up this portfolio as much as we wanted to make as large a profit as we desired.

(b) Law of one price

The law of one price states that two portfolios that behaves in exactly the same way must have the same price. If this were not true, we could buy the 'cheap' one and sell the 'expensive' one to make an arbitrage (risk-free) profit.

(c)

i. Using Put call parity, the value of put option should be:

$$P_t = C_t + K\exp(-r(T-t)) - S_t\exp(-q(T-t))$$

$$= 30 + 120\exp(-.05 \times 0.25) - 125\exp(-0.15 \times 0.25)$$

$$= 28.11$$

ii. Arbitrage profit

If the put options are only Rs 23 then they are cheap. If things are cheap then we buy them. So looking at the put-call parity relationship; we "buy the cheap side and sell the expensive side" ie. we buy the put options and shares and sell call options and cash. 2.

For example:

- Sell 1 call option (Rs 30)
- buy 1 put option (₹23)
- buy 1 share (₹125)
- sell (borrow) cash ₹118

This is a zero-cost portfolio and, because put-call parity does not hold, we know it will make an arbitrage profit. We can check as follows:

In 3 months time, repaying the cash will cost us $118\exp(0.05 \times 3/12)$ = Rs 119.48

We also receive dividends d on the share:

1. If the share price is above 120 in 3 months' time then the other party will exercise their call option

and we will have to give them the share. They will pay Rs 120 for it and our profit is

$120 - 119.48 + d = 0.52 + d$

Therefore, the put option is useless to us

2. If the share price is below 120 in 3 months' time then we'll expire our put option and sell it for 120.

Our profit is

$120 - 119.48 + d = 0.52 + d$

2. Let $dX_t = A_t \, dt + B_t \, dZ_t$

where, $A_t = \alpha \mu (T-t)$, $B_t = \sigma \sqrt{T-t}$ —(i)

$$dF = \frac{\delta f}{\delta x} B_t dZ_t + \left(\frac{\delta f}{\delta t} + \frac{\delta f}{\delta x} A_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} B_t \right) dt$$

$$= -\int B_t \, dZ_t + \int \frac{\partial m}{\partial t} (T-t) \, dt - \int A_t \, dt + \frac{1}{2} \int B_t^2 \, dt$$

$$dF = \int \left(\frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 \right) dt - \int B_t \, dZ_t$$

For F to be martingale, $\frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 = 0$

Thus $\frac{\partial m}{\partial t} (T-t) = A_t - \frac{1}{2} B_t^2$

Substituting,

$$\frac{\partial m}{\partial t} = \alpha \mu - \frac{1}{2} \sigma^2$$

3. $Y_t = f(t, X_t)$

Applying Ito's lemma

$$dY_t = \frac{df}{dt}dt + \frac{df}{dx}dX_t + \frac{1}{2}\frac{d^2f}{dx^2}\sigma^2 x_t^2 \, dt$$

$$= Y_t \, dt + 2e^{t} X_t^2 \frac{dX_t}{X_t} + e^{-t}\sigma^2 X_t^2 \, dt$$

$$= -Y dt + 2Y_t (0.25 dt + \sigma \, dW_t) + \sigma^2 Y_t \, dt$$

$$\frac{dY_t}{Y_t} = (2 \times (0.25) - 1 + \sigma^2) dt + 2\sigma \, dW_t$$

$$= (\sigma^2 - 0.5) dt + 2\sigma \, dW_t$$

$$\therefore dY_t = (\sigma^2 - 0.5) Y_t \, dt + 2\sigma Y_t \, dW_t$$

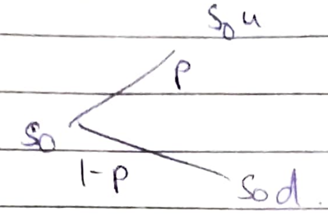
4. Let n ex-dividend dates are anticipated for a stock and $t_1 < t_2 < \cdots < t_n$ are the times before which the stock goes ex-dividend. Dividends are denoted by $q_1 \cdots d_n$.
If the option is exercised prior to the ex-dividend date then the investor receives $S(t_n) - K$.
If the option is not exercised, the price drops the
The value of the american option is greater than $S(t_n) - d_n$.
It is never optimal to exercise the option if.
$d_n <= K \times (1 - \exp(-r(T - t_n)))$
Using this equation: we have $K \times (1 - \exp(-r(T - t_n)))$
$= 350 \times (1 - \exp(-0.95(0.833 - 0.25))) = 18.87$ and similarly 10.91. Hence, it is never optimal to exercise the american option on the two ex-dividend dates.

5. The required probability is the probability of the stock price being greater than ₹ 258 in 6 months time.
The stock price follows geometric Brownian motion
ie. $S_t = S_0 \exp(\mu - \sigma^2/2)t + \sigma W_t$
∴ $\ln(S_t)$ follows normal distribution with mean $\ln(S_0) + (\mu - \sigma^2/2)t$ and variance $(\sigma^2)t$.
Implies $\ln(S_t)$ follows $\varphi(\ln 254 + (0.16 - 0.35^2/2) \times 0.5, 0.35 \times 0.5^{(1/2)})$
$= \varphi(5.59, 0.247)$

This means $[\ln(S_t) - S_t)]/\sigma t^{(1/2)}$ follows standard Normal
Hence, the probability that stock price will be higher than the strike price of Rs 258 in 6 months time =
$1 - N(5.55 - 5.59)/0.247 = 1 - N(-0.1364) = 0.5542$

6. (i) The given relationship can be written as
$S_t = S_0 e^{\mu t + \sigma B_t}$
∵ S_t is a function of standard Brownian motion, B_t, applying Ito's lemma, the SDE for the underlying stochastic process becomes:
$dB_t = 0 \times dt + 1 \times dB_t$
Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B_t}$, then
$dG/dB_t = \sigma S_0 e^{\mu t + \sigma B_t} = \sigma S_t$
$d^2G/dB_t^2 = \sigma^2 S_0 e^{\mu t + \sigma B_t} = \sigma^2 S_t$
Hence using Ito's lemma from Pg 46 in tables
$dG = [0 \times \sigma S_t + 1/2 \times 1^2 \times \sigma^2 S_t + \mu S_t] dt + 1 \times \sigma S_t dB_t$
ie. $dS_t = (\mu + 1/2\sigma^2) S_t dt + \sigma S_t dB_t$
So, $C_1 = \sigma$ & $C_2 = \mu + 1/2\sigma^2$

7. (i) Setting up the commodity tree using u for up move and d for down move p is up-step probability



Where p is the up probability & (1-p) the down probability

Then $E(C_t) = S_0 [pu + (1-p)d]$ and

$Var(C_t) = E(C_t^2) - E(C_t)^2$

$= S_0^2 \, p(1-p)(u-d)^2$

Equating moments.

$S_0 e = S_0 [pu + (1-p)d]$ — (A)

And $\sigma^2 S_0^2 t = S_0^2 \, p(1-p)(u-d)^2$ — (B)

From (A) we get.

$$p = \frac{e^{rt} - d}{u-d} \quad \text{— (C)}$$

Substituting

$$\sigma^2 t = \frac{e^{rt} - d}{u-d}\left(1 - \frac{e^{rt} - d}{u-d}\right)(u-d)^2$$

Putting $d = 1/u$ & multiplying through by u we get

$$u^2 e^{rt} - u(1 + e^{2rt} + \sigma^2 t) + e^{rt} = 0$$

15. i Let $f = f(S_t, t) = S^k$

$$\frac{\partial f}{\partial S} = kS^{k-1}, \quad \frac{\partial^2 f}{\partial S^2} = k(k-1)S^{k-1}, \quad \frac{\partial d}{\partial t} = 0$$

Using Ito's calculus, $df = kS^{k-1}dS_t + 1/2\, k(k-1)S^{k-2}(dS_t)^2$
$$= f[k\mu + 1/2\, k(k-1)\sigma^2]dt + f(k\sigma)dW_t$$

Hence, $f = S^k$ follows Geometric Browian motion with drift
Hence $f_1 = f_0 e^{\left(\mu t - 1/2\, e^{1/2}\right)t - \sigma W_t}$.
This means $\frac{f}{f_0} \sim$ log normal $((\mu' - 1/2\, \sigma'^2)t, \sigma^2 t)$

The mean & variance of a log normal distribution with parameters μ & σ^2 is given by $e^{\mu + 1/2\sigma^2}$ & $(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$

This means $E(f) = f_0 e^{\mu t}$ & $V(f) = f_0^2 e^{2\mu t}(e^{\sigma'^2 t} - 1)$

ii Let $f = f(t, S_t) = e^{-rt} S_t$

$$\frac{\partial f}{\partial t} = -re^{-rt}S_t, \quad \frac{\partial f}{\partial S} = e^{-rt}, \quad \frac{\partial^2 f}{\partial S^2} = 0$$

By Ito calculus, it is a martingale if & only if $\mu = r$.

iii. Combining results from part (a) & (b), we need, for discounted S^k to be a martingale,
$$k\mu + 1/2\, k(k-1)\sigma^2 = r$$
For given values of r, σ & k one can solve for value of μ for which discounted S^k will be martingale.

13.

i. Let S_t / S_0 follows log normal distribution with parameters $\left[\mu - \frac{1}{2}\sigma^2\right]t$ & $\sigma^2 t$ such the expected return on a stock is μ & volatility is σ.

This mean Expected value of stock price at the end of first time step $= S_0 e^{\mu \delta t}$. On the tree expected price at that time $= q S_0 u + (1-q) S_0 d$

In order to match the expected return on the stock with the tree's parameters we have $q S_0 u + (1-q) S_0 d = S_0 e^{\mu \delta t}$

Volatility σ of a stock price is defined so that $\sigma \sqrt{\delta t}$ is the standard deviation of the return on the stock price in a short period of time δt.

The variance of stock price return is

$$q u^2 + (1-q) d^2 - [qu + (1-q) d]^2 = \sigma^2 \delta t$$

substituting

$$e^{\mu \delta t}(u + d) - ud - e^{2\mu \delta t} = \sigma^2 \delta t$$

when higher powers of δt other than δt are ignored.

This implies $u = e^{\sigma \sqrt{\delta t}}$ & $d = 1/u = e^{-\sigma \sqrt{\delta t}}$

ii.

S	200
r	10%
σ	35%
T	2 months
t	1 month = 0.0833
u	1.1063
d	[illegible]
q	0.516
$p(1-q)$	[illegible]
k	[illegible]

0 [illegible]

0

200

200.00

173 [illegible]

6.62

190.15

9.85

165.41

180 [illegible]

The value of option at time 1 is

$(6.62 \times q + 19.77(1-q))e^{-0.10/12} = 12.61$

This is more than the payoff at time 1 hence the American option will not be exercised

The value of put option at time 0 is

$12.61 \times e^{-0.1/12}(1-q) = 6.05$

14.

(i) Pay off Diagram for European call:

		51.8	u	1.06
	20.000	561.8	d	0.95
	530.000		r	0.05
500		503.5	t	3 months = 0.25
	475		p	(exp(r × t − d)/(u−d) =
	0	451.25	~~p~~	0.5689
		0	1−p	0.4311

Value of the 6 month European call = exp(−0.5 × 0.05) ×
51.8 × 0.5689^2
= 16.351

(ii) The pay off diagram for European put.

	B	O
	0	561.8
	530	
500		503.56.50
	475	
	35	451.25
C		58.75

Value of European put option:
exp(0.05 × 0.5) × [0.4311^2 × 58.75 + 2 × 0.4311 × 0.5689 × 6.05]
= 13.759

Using Put call parity. value of put + stock = 500 + 13.759 = 513.79
value of call + discounted price = 16.351 + 510 × exp(−0.05 × 0.5)
= 53.759

iii. For the American option, the value at final nodes is same for a European Option. At earlier nodes, the value of the option is greater of value calculated at the node & the payoff from early exercise.

We can find that the value at node B

$= 6.50 \times \exp(-0.25 \times 0.05) \times 0.4311 = 2.767$

Value at node C $= \exp(-0.25 \times 0.05) \times (6.30 \times 0.5689 + 30.75 \times 0.4311) = 2\ldots$

However, payoff from immediate exercise is 35 at node C. Hence it is advisable to exercise the put option at this point.

Value of the American put option at time t = 0 will be 16.46.

iv. Expected payoff in 3 months time is calculated by using real rate of return of 9%. p = 0.6614. Hence expected payoff = 13.228. Unfortunately, it is not easy to know the correct discount rate to apply to the expected payoff in real world to be able to compute the value of the option. Risk neutral valuation solves this problem as under risk neutral valuation all assets are expected to earn the risk free rate.