Tanishq

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## Assignment-1

(1) $$\frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\lambda}{n}}} \sim N(0,1)$$

$$\hat{\lambda} = \frac{\sum x_i}{n}$$

$$\hat{\lambda} = 200$$

$$P(Z_{2.5} < Z < Z_{97.5}) = 0.95$$

$$P(-1.96 < Z < 1.96) = 0.95$$

$$P\left(-1.96 < \frac{\hat{\lambda} - \lambda}{\sqrt{\frac{\hat{\lambda}}{n}}} < 1.96\right) = 0.95$$

$$P\left(-1.96\sqrt{\frac{\hat{\lambda}}{n}} < \hat{\lambda} - \lambda < 1.96\sqrt{\frac{\hat{\lambda}}{n}}\right) = 0.95$$

$$P\left(\lambda - 1.96\sqrt{\frac{\hat{\lambda}}{n}} < \lambda < \hat{\lambda} + 1.96\sqrt{\frac{\hat{\lambda}}{n}}\right) = 0.95$$

95% confidence interval of $\lambda$ when $\hat{\lambda} = 200$ & $n = 1$

95% CI for $\lambda = (200 - 1.96\sqrt{200},\ 200 + 1.96\sqrt{200})$

$= (172.28141,\ 227.7185)$

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(i) $x_i$, $i = 1, 2 \ldots n$ are i.i.d

Mean $= \mu$

Variance $= \sigma^2$

$$S^2 = \frac{1}{n-1}\left(\Sigma x_i^2 - n\bar{x}^2\right)$$

i) $$E(\bar{x}) = E\left(\frac{\Sigma x_i}{n}\right)$$

$$= \cancel{E} \frac{\Sigma E(x_i)}{n}$$

$$= \frac{\cancel{n}\mu}{\cancel{n}}$$

$$= \mu$$

$$V(\bar{x}) = V\left(\frac{\Sigma x_i}{n}\right)$$

$$= \frac{1}{n^2} V(\Sigma x_i)$$

$$= \frac{1}{n^2} \Sigma V(x_i)$$

$$= \frac{\cancel{n}\sigma^2}{\cancel{n^2}}$$

$$= \frac{\sigma^2}{n}$$

(ii) To show

$$E(s^2) = \sigma^2$$

$$E(s^2) = E\left[\frac{1}{n-1}\left(\sum x_i^2 - n\bar{x}^2\right)\right]$$

$$= \frac{1}{n-1}\left[E\left(\sum x_i^2\right) - nE(\bar{x}^2)\right]$$

$$= \frac{1}{n-1}\left[\sum E(x_i^2) - nE(\bar{x}^2)\right]$$

Now,

$$V(X) = E(x_i^2) - [E(X)]^2$$

$$E(\bar{x}^2) = \mu^2 + \frac{\sigma^2}{n}$$

$$E(s^2) = \frac{1}{n-1}\left(\sum \mu^2 + \sigma^2 - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right)$$

$$= \frac{1}{n-1}\left(n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2\right)$$

$$= \frac{1}{(n-1)} \times \sigma^2(n-1)$$

$$E(s^2) = \sigma^2$$

(iii) Now: $X \sim N(\mu, \sigma^2)$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$V\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{(n-1)}{\sigma^4} V(S^2)$$

$$V\left(\chi^2_{n-1}\right) = \frac{(n-1)}{\sigma^4} V(S^2)$$

$$2\cancel{(n-1)} = \frac{(n-1)^{\cancel{2}}}{\sigma^4} V(S^2)$$

$$V(S^2) = \frac{2\sigma^4}{(n-1)}$$

(iv) We need,

$$P(0.5\sigma^2 < S^2 < 1.5\sigma^2)$$

$$= P(S^2 < 1.5\sigma^2) - P(S^2 < 0.5\sigma^2)$$

$n = 10$

$\sigma^2 = 100$

$$P(S^2 < 1.5\sigma^2) = P(S^2 < 150)$$

$$= P\left(\frac{9S^2}{\sigma^2} < \frac{150 \times 9}{100}\right)$$

$$= P\left(\chi^2_9 < 13.5\right)$$

$$= 0.8587$$

(iv) $P(s^2 < 0.5\sigma^2) = P(s^2 < 50)$

$$= P\left(\frac{9s^2}{\sigma^2} < \frac{50\times 9}{100}\right)$$

$$= P\left(\chi^2_9 < 4.5\right)$$

$$= 0.1245$$

$P(50 < s^2 < 150) = 0.8587 - 0.1245$

$= 0.7342$

(3) Let nos of claims be a random variable $X$

$X \sim (4, p)$

(i) $$L = \prod_{i=1}^{186} f(x_i)$$

$$= (P(X=0))^{86} \times (P(X=1))^{75} \times [P(X=2)]^{16} \times [P(X=3)]^{2} \times [P(X=4)]^{1}$$

$$= ({}^4C_0\, p^0 (1-p)^4)^{86} \times ({}^4C_1\, p^1 (1-p^2)^{75}) \times ({}^4C_2\, p^2 (1-p)^2)^{16} \times ({}^4C_3\, p^3 (1-p))^2 \times ({}^4C_4\, p^4)^1$$

$$= \text{constant} \times p^{117} \times (1-p)^{603}$$

$\log(L) = \log(\text{constant}) + 117\log(p) + 603(1-p)$

$$= 0 + \frac{117}{p} - \frac{603}{1-p}$$

$$\frac{d L(p)}{d p} = 0$$

$$\frac{117}{p} = \frac{603}{1-p}$$

$$117 - 117p = 603p$$

$$720p = 117$$

$$\hat{p} = \frac{117}{720} \qquad \hat{p} = 0.1625$$

[illegible] Goodness of fit test

$H_0$: no. of claims confirms to the binomial model.

$H_1$: no. of claims confirm to the binomial model.

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $O_i$ | 86 | 75 | 16 | 2 | 1 |
| $E_i$ | 88.54 | 68.724 | 19.998 | 2.574 | 0.108 |

$p = 0.1625$

$$P(X=0) = {}^4C_0\, p^0 (1-p)^4 = 0.4919$$

$$P(X=1) = {}^4C_1\, p^1 (1-p)^3 = 0.3818$$

$P(X=2) = 0.111$

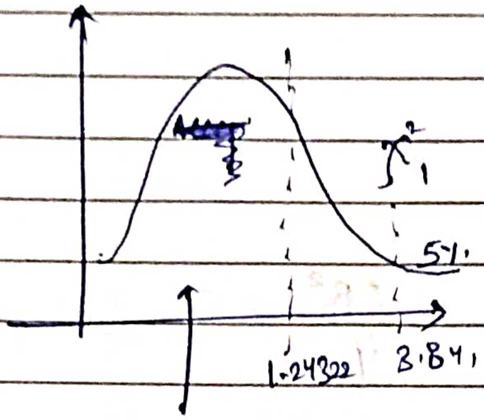
$P(X=3) = 0.0143$

$P(X=4) = 0.0006$

| | | | |
|---|---|---|---|
| $D_i$ | 86 | 75 | 19 |
| $E_i$ | 88.542 | 68.724 | 22.68 |

$$T.S = \sum \frac{(D_i - E_i)^2}{E_i} \sim \chi^2_{5-1-1-2}$$

$$1.24322 \sim \chi^2_1$$




Acceptance region

we have insufficient evidence to reject $H_0$ @ 5% level of significance

Therefore, nos. of claims confirms to the binomial model.

(4) $f(x) = \frac{CB^3}{(x+\beta)^4}$ ; $x > 0, \beta > 0$

i) Comparing the above $f(x)$ to PDF of pareto distribution (2 parameter version)

For pareto,

$$PDF = \frac{\alpha \lambda^{\alpha}}{(\lambda+x)^{\alpha+1}}$$

This $\beta$ corresponds to $\alpha$

$\Rightarrow$ Thus, $c=3$ (as $c$ corresponds to $\alpha$ here)

$$E(X) = \frac{\lambda}{\alpha-1}$$

$$E(X) = \frac{\beta}{2}$$

$$V(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = \frac{3\beta^2}{4}$$

i) $E(X) = \frac{\beta}{2}$ $\qquad$ $E(X) = \bar{X}$

$\frac{\beta}{2} = \bar{X}$ $\qquad$ $\beta = 2\bar{X}$

$$\Rightarrow \text{Bias} = E(\hat{\beta}) - \beta$$
$$= E(2\bar{X}) - \beta$$
$$= 2E(\bar{X}) - \beta$$
$$= \frac{2}{n}\sum E(X_i) - \beta$$
$$= \beta - \beta$$

Bias $= 0$, here unbiased.

$\Rightarrow$ MSE $= V(\hat{\beta}) +$ Bias

$$= \frac{2}{n} \times \frac{n\beta}{2} - \beta$$

Bias $= 0$, hence unbiased

$$MSE = V(\hat{\beta}) + (Bias)^2$$

$$= V(\hat{\beta})$$

$$= V(2\bar{x})$$

$$= \frac{4}{n^2} \sum V(X_i)$$

$$= \frac{\cancel{4}n}{\cancel{n^2}_{\,n}} \times \frac{3\beta^2 \cancel{n}}{\cancel{4}}$$

$$MSE = \frac{3\beta^2}{n}$$

As $n$ increases, MSE tends to 0

Hence $\beta$ is unbiased.

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iii) $\hat{\beta} = b\bar{X}$

$E(\hat{\beta}_2) = E(b\bar{X})$

$= b\,E\left(\frac{\sum X_i}{n}\right)$

$= \frac{b}{\cancel{n}} \times \frac{\beta}{2} \times \cancel{n}$

$= \frac{b\beta}{2}$

$V(\hat{\beta}_2) = V(b\bar{X})$

$= b^2 V\left(\frac{\sum X_i}{n}\right)$

$= \frac{b^2}{n^2} \sum V(X_i)$

$= \frac{b^2}{\cancel{n^2}\,n} \times \frac{3}{4} \times \beta^2 \times \cancel{n}$

$= \frac{3b^2\beta^2}{4n}$

$MSE = V(\beta_2) + (Bias)^2$

$= \frac{3b^2\beta^2}{4n} + \left(\frac{b\beta}{2} - \beta\right)^2$

~~MSE = V(β) + (Bias)²~~ $= \frac{3b^2\beta^2}{4n} + \left(\frac{b^2\beta^2 + 4\beta^2 - 4\beta^2}{4}\right.$

$= \frac{3b^2\beta^2 + nb^2\beta^2 + 4n\beta^2 - 4n\beta^2 b}{4n}$

$MM = \frac{\beta^2}{4n}\left(3b^2 + nb^2 + 4n - 4nb\right.$

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$$\frac{d(M)}{db} = \frac{\beta^2}{4n}(6b + 2nb - 4n)$$

$$\frac{d(M)}{d(b)} = 0$$

$$\frac{\beta^2}{4n}(6b + 2nb - 4n) = 0$$

$$6b + 2nb - 4n = 0$$

$$3b + nb - 2n = 0$$

$$b = \frac{2n}{3+n}$$

$$b = \frac{2n}{3+n}$$

$$b = \frac{2}{1+\frac{3}{n}}$$

$$\frac{d^2M}{db^2} = \frac{\beta^2}{4n}(6+2n)$$

$$\frac{\beta^2}{4n}(6+2n) > 0$$

$b = \frac{2}{1+3/n}$ mininimum with MLE

(iv) $E(\beta_2) = \frac{b\beta}{2}$

$$= \frac{2}{1+3/n} \times \beta$$

$$= \frac{n\beta}{n+3}$$

$$\text{Bias} = E(\hat{\beta}_2) - \beta$$

$$= \frac{n\beta - \beta(n+3)}{n+3}$$

$$= \frac{n\beta - n\beta - 3\beta}{n+3}$$

It is a negative bias, & hence an underestimate of true parameter $\beta$.

$$MLE = \frac{\beta^2}{4n}(3b^2 + nb^2 + 4n - 4nb)$$

As $n \to \infty$, M.S.E does not tend to 0, hence inconsistent

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(5) (i) $X \sim U(a,b)$

$$E(X) = \frac{a+b}{2}$$

$$\bar{x} = \frac{a+b}{2}$$

$$a = 2\bar{x} - b \quad (1)$$

$$V(X) = \frac{(b-a)^2}{12}$$

$$S^2 = \frac{(b-a)^2}{12}$$

$$12S^2 = (b-a)^2$$

$$2\sqrt{3}S = b-a.$$

$$2\bar{X} - 2a = 2\sqrt{3}S.$$

$$\sqrt{3}S = \bar{x} - a$$

(ii) $b = 2\bar{x} - a$

$$b = 2\bar{x} - \bar{x} + \sqrt{3}S$$

$$\hat{b} = \bar{X} + \sqrt{3}S$$

(iii) Sample of derivatives

1, 2, 8, 50, 150

$$\bar{x} = 42.2$$

$$S = 63.57$$

$$\hat{a} = \bar{X} - \sqrt{3}S.$$

$$= -67.906$$

$$\hat{b} = \bar{X} + \sqrt{3}\, S.$$

$$= 152.364$$

Hence we can conclude the value of $\hat{b}$ as approx very close to sample value but $\hat{a}$ is very far from sample.