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Probability and Statistics Assignment - 2

1. Let X be the amount of a home insurance claim and Y the amount of a car insurance claim.

$X \sim N(800, 100^2)$ $\qquad Y \sim N(1200, 300^2)$

we require,

$P[(Y_1 + Y_2 + Y_3)] > [(X_1 + X_2 + X_3 + X_4) + 800]$

$= P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$

we need distribution of $(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4)$

$= \sim N(400, 31000)$

$= P\left(Z > \frac{800 - 400}{\sqrt{31000}}\right) - \cancel{P(Z > 0.8718)}$

$= 1 - P(Z < 0.718)$

$= 0.236$

3 (i) $f(x) = \lambda e^{-\lambda(x-\alpha)}, \quad x \geq \alpha \quad \lambda, \alpha > 0$

using MGF distribution

$$M_X(t) = E(e^{tX}) = \int_{\alpha}^{\infty} e^{tX} \cdot \lambda e^{-\lambda(x-\alpha)} \cdot dx$$

$$= \int_{\alpha}^{\infty} \lambda e^{\lambda\alpha} e^{-(\alpha - t)x} \cdot dx$$

$$= \left[-\frac{\lambda e^{\lambda\alpha}}{\lambda - t} \cdot e^{-(\lambda-t)x} \right]_{\alpha}^{\infty}$$

$= \frac{\lambda}{\lambda - t} \cdot e^{t\alpha}$, provided $t < \lambda$

$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$

$M'_x(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-2} \cdot e^{t\alpha} + \alpha\left(1 - \frac{t}{\lambda}\right)^{-1} \cdot e^{t\alpha}$

$E(x) = M'_x(0) = \frac{1}{\lambda} + \alpha$

$M''_x(t) = \frac{2}{\lambda^2}\left(1 - \frac{t}{\lambda}\right)^{-3} e^{t\alpha} + \frac{2\alpha}{\lambda}\left(1 - \frac{t}{\lambda}\right)$
$+ \alpha^2 \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$

$E(x^2) = M''_x(0) = \frac{2}{\lambda^2} + \frac{2}{\lambda}\alpha + \alpha^2$

$Var(X) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left(\frac{1}{\lambda} + \alpha\right)$

$Var(x) = \frac{1}{\lambda^2}$

Prob, $P(X = n) = \frac{1}{n!} G_X^{(n)}(0)$

$P(V = 4) = \frac{1}{4!} G_V^{4}(0)$

$G'_V(t) = k p^k q (1 - qx)^{-k-1}$

4. $f(x,y) = ax(x+y)$, $0 < x < 5$

we know that, $10 < y < 20$

$$\int_{10}^{20}\int_{0}^{5} ax(x+y)\,dx\,dy = 1$$

$$\int_{10}^{20}\int_{0}^{5} a(x^2+xy)\,dx\,dy = 1$$

$$\int_{10}^{20} a\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5 dy = 1$$

$$\int_{10}^{20} a \cdot \left[\frac{125}{3} + \frac{25}{2}\right] y \cdot dy = 1$$

$$a\left[\frac{125}{3}y + \frac{25}{4}y^2\right]_{10}^{20} = 1$$

$$a \cdot \left[\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right] = 1$$

$$a(2291.666) = 1$$

$$a = 0.00044$$

$$E(Y/x) = \int_y y \cdot \frac{f(x,y)}{f(x)} \cdot dy$$

$$f(x) = \int_y f(x,y) \cdot dy$$

$$= \int_{10}^{20} a \cdot (x^2+xy)\,dy$$

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$$= a \cdot \left[x^2 y + \frac{xy^2}{2} \right]_{10}^{20}$$

$$= a \left[20x^2 + 200x - 10x^2 - 50x \right]$$

$$= a[10x^2 + 150x]$$

$$E(Y/X) = \int_{10}^{20} y \, \frac{a[x^2 + xy]}{a[10x^2 + 150x]} \, dy$$

$$= \int_{10}^{20} y \cdot \frac{(x+y)}{(10x+150)} \, dy$$

$$= \frac{1}{10x+150} \left[\frac{y^2}{2} x + \frac{y^3}{3} \right]_{10}^{20}$$

$$= \frac{1}{10x+150} \left(200x + \frac{8000}{3} - 50x - \frac{1000}{3} \right)$$

$$= \frac{1}{10x+150} \left(150x + \frac{7000}{3} \right)$$

Examining Arthur's Statements :

$X \sim Poi(\lambda)$ & $Y \sim Poi(\mu)$

X has MGF

$$M_X(t) = e^{\lambda(e^t - 1)}$$

&

Y has MGF

$$M_Y(t) = e^{\mu(e^t - 1)}$$

So the difference is,

$(X-Y)$ has MGF

$$M_X(t) \cdot M_Y(t) = e^{\lambda(e^t-1)} \, e^{-\mu(e^t-1)}$$
$$= e^{(\lambda-\mu)(e^t-1)}$$

which is the MGF of a Poisson with parameters $\lambda - \mu$

So Arthur's statement is true

similarly,

$X \sim Poi(\lambda)$ & $Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ & $M_Y(t) = e^{\mu(e^t-1)}$

so the sum $X+Y$ has MGF,

$$M_X(t) \, M_Y(t) = e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$$
$$= e^{(\lambda+\mu)(e^t-1)}$$

→ MGF of Poisson with parameter $\lambda+\mu$,

→ $X+Y \sim Poi(\lambda+\mu)$

∴ They're statement is also True.

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(i) $var\left(X/Y=2\right) = E\left(X^2/Y=2\right) - E^2\left(X/Y=2\right)$

$E(X/Y=2) = 1 \cdot \frac{0}{0.6} + 2\times\frac{0.3}{0.6} + 3\times\frac{0.1}{0.6} + 4\times\frac{0.2}{0.6}$

$= \frac{17}{6}$

$E(x^2/Y=2) = 1^2\times\frac{0}{0.6} + 2^2\times\frac{0.3}{0.6} + 3^2\times\frac{0.1}{0.6} + 4^2\times\frac{0.2}{0.6}$

$= \frac{53}{6}$

so $var(X/Y=r) = \frac{53}{6} - \left(\frac{12}{6}\right)^2$

$= \frac{29}{36} = 0.80556$

Conditional Expectation

$E(u/v=v) = \int_u u f(u/v)\, du$

$f(v) = \int_{u=0}^{1} \frac{48}{67}(2uv - u^2)\, du$

$= \frac{48}{67}\left[u^2v - \frac{u^3}{3}\right]_0^1$

$= \frac{48}{67}\left[v - \frac{1}{3}\right]$

$\Rightarrow f(u/v) = \frac{f(u,v)}{f(v)} = \frac{\cancel{\frac{48}{67}}(2uv - v^2)}{\cancel{\frac{48}{67}}\left(v-\frac{1}{3}\right)}$

$= \frac{(2uv - v^2)}{\left(v - \frac{1}{3}\right)}$

$\rightarrow$

so, $E(U/V=v) = \int_0^{1} \frac{2u^2v - u^3}{v - \frac{1}{3}} du$

$= \left[\frac{2/3\, u^3 v - \frac{u^4}{4}}{v - \frac{1}{3}} \right]_0^1$

$= \frac{\frac{2}{3}v - \frac{1}{4}}{v - \frac{1}{3}}$

8. $X \sim Gamma(3,2)$

$E(Y/X=x) = 3x+1$

$var(Y/X=x) = 2x^2+5$

$E(Y) = E(E(Y/X)) = E(3x+1)$

$= 3E(x)+1$

but $E(x) = \frac{3}{2} = 1.5$

$E(Y) = 3(1.5)+1$

$= 4.5+1$

$E(Y) = 5.5$

using $var(Y) = E(var(Y/X)) + var[E(Y/X)]$

$= var(Y) = E(2x^2+5) + var(3x+1)$

$= 2E[x^2] + 5 + 9\,var(x)$

$= 2(1.5)^2 + \left(\frac{9}{16}\right) + 9\left(\frac{3}{4}\right)$

$= 2(2.8125) + 6.75$

$var(Y) = 12.375$

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7. $E(x) = 3$

$sdx = 2$

$E(Y) = 4$

$sdy = 1$

Correlation coeffecient $= -0.3$

$z = x + y$

(i) $\text{cov}(x, z) = \text{cov}(x, x + y)$

$= \text{cov}(x, x) + \text{cov}(x, y)$

$= \text{var}(x) + \text{cov}(x, y)$

using correlation coeffecient between x & y given:

$$\text{cov}(x, y) = -0.3 = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x)\,\text{var}(y)}} = \frac{\text{cov}(x, y)}{\sqrt{4 \times 1}}$$

$\text{cov}(x, y) = -0.6$

$\text{cov}(x, z) = 4 - 0.6 = 3.4$

(ii) $\text{var}(z) = \text{cov}(z, z)$

$\text{var}(z) = \text{cov}(x + y, x + y)$

$= \text{cov}(x, x) + 2\text{cov}(x, y) + \text{cov}(y, y)$

$= \text{var}(x) + 2\text{cov}(x, y) + \text{var}(y)$

$= 4 + 2(-0.6)$

$= 3.8$

10. $kx^{-\alpha} e^{y/\beta}$ $\quad 1 < x < \infty$

$1 < y < \infty$

since the pdf must integrate to 1

$$\int_{y=1}^{\infty} \int_{x=1}^{\infty} kx^{-\alpha} e^{y/\beta} \, dx \cdot dy = 1$$

$$= \int_{1}^{\infty} kx^{-\alpha} e^{-y/\beta} = ke^{-y/\beta} \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_{1}^{\infty}$$

$$= \int_{1}^{\infty} \frac{ke^{-y/\beta}}{\alpha-1} = \frac{k}{\alpha-1} \left[-\beta e^{-y/\beta} \right]_{1}^{\infty}$$

$$= \frac{k \cdot \beta e^{-1/\beta}}{\alpha-1}$$

Equating it to 1

$$\frac{k\beta e^{-1/\beta}}{\alpha-1} = 1$$

$$k = \frac{(\alpha-1) e^{1/\beta}}{\beta}$$