ASSIGNMENT 1 - SRM

Q1

i) A **Stochastic Model** has the capacity to handle uncertainties in the inputs applied. Stochastic models possess some inherent randomness - the same set of parameter values and initial conditions will lead to an ensemble of different outputs.

Advantages of stochastic models:

1) Stochastic models can reflect real-world economic scenarios that provide a range of possible outcomes you may experience and the relative likelihood of each.
2) By running thousands of calculations, using many different estimates of future economic conditions, stochastic models predict a range of possible future investment results showing the potential upside and downsides of each.

ii)

The time spent in state H before the next visit to S has mean σ ^(-1)

Therefore a reasonable estimate for σ is the reciprocal of the mean length of each visit: = (Number of transitions from H to S)/(Total time spent in state H up until the last transition from H to S), although it would be equally valid to use the Maximum Likelihood Estimator, which is (Number of transitions from H to S)/(Total time spent in state H).

Similarly for ρ .

Q2

i) Consider a Markov chain taking values in the set S = {i : i = 0, 1, 2, 3, 4}, where i represents the number of umbrellas in the place where the Actuary currently is (at home or office).

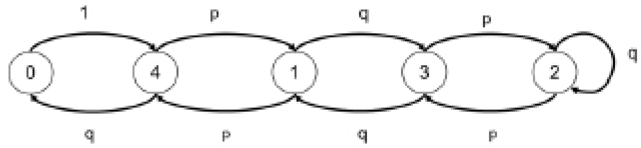
If i = 1 and it rains then he take the umbrella, move to the other place, where there are already 3 umbrellas, and, including the one he brings, therefore he will now have 4 umbrellas. Thus, p1,4 = p, because p is the probability of rain.

If i = 1 but does not rain then he do not take the umbrella, goes to the other place and find 3 umbrellas. Thus, p1,3 = 1 − p ≡ q.

However, if i=0, he must move to other place where 4 umbrellas are kept with probability

P0,4 =1

Similarly for other states. The process is depicted by the following diagram.



ii)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q & p \\ 0 & 0 & q & p & 0 \\ 0 & q & p & 0 & 0 \\ q & p & 0 & 0 & 0 \end{pmatrix}$$

iii) For stationary distribution πP = π

Writing the equations:

π (0) = π (4).q ----1

π (1) = π (3).q + π (4).p ----2

π (2) = π (2).q + π (3).p ----3

π (3) = π (1).q + π (2).p ----4

π (4) = π (0) + π (1).p ----5

Substituting 1 in 5

π (4) =π (4).q + π (1).p => π (1) = π (4). (1-q)/ p

=> π (1) = π (4). p/ p => π (1) = π (4)

Substituting 6 in 2

π (1) = π (3).q + π (4).p => π (4) = π (3).q + π (4).p

=> π (3) = π (4).(1-p)/q => π (3) = π (4) ---7

Substituting 7 in 3

π (2) = π (2).q + π (3).p => π (2) = π (3).p/(1-q)

=> π (2) = π (3).p/(1-q) => π (2) = π (3). ----8

Substituting 6 & 7 in 4

π (3) = π (1).q + π (2).p => π (4) = π (1).q + π (4).p

=> π (1) = π (4).(1-p)/q => π (1) = π (4)

Therefore π (1)= π (2)= π (3)= π (4) = π (0)/q

But π (0) + π (1) + π (2) + π (3) + π (4) = 1, substituting in terms of π (4)

π (4).q + 4 π (4) =1

Therefore π (4) =1/(4+q) = π (1)= π (2)= π (3) and π (0) =q/(4+q)

iv) He will wet every time he happens to be in state 0 and it rains. The chance he is in state 0 is π (0). The chance it rains is p. Hence Probability that he gets wet.
P(WET) = Probability of being in state 0 and its raining
P(WET) =(4/(4+q))p

v) P(WET) = 0.6*0.4/(4+0.4) = 5.45%
If he want the chance to be less than 1% then, clearly, he need more umbrellas. So, suppose he need N umbrellas. Set up the Markov chain as above and generalising, it is clear that
π (1)= π (2)= π (3)= π (4) …… = π (N) = π (0)/q.
But π (0) + π (1) + π (2) + π (3) + π (4) … π (N) = 1, substituting in terms of π N)
=> π (N).q + N π (N) =1
=> π (N) =
Therefore probability of getting Wet P(WET)= p* π (0) =
For P(WET) < 1/100 =>pq/(N+q) < 1/100
N > 100*p*q – q ; given p = 0.6, q =0.4

Q3

i), ii)

Process	State Space	Time Domain
Counting Process	Discrete	Discrete or Continuous
General Random Walk	Discrete or Continuous	Discrete
Poisson Process	Discrete	Continuous
Markov Chain	Discrete	Discrete
Markov Jump	Discrete	Continuous

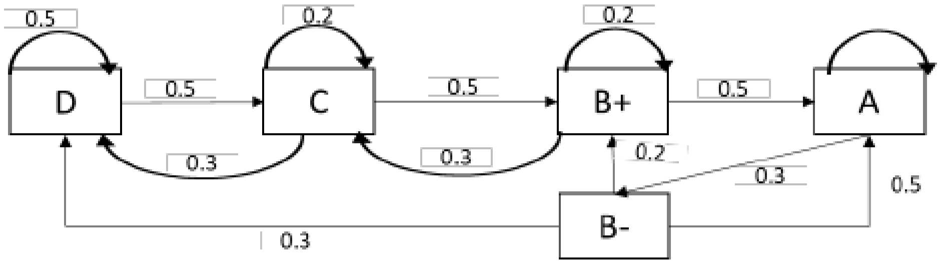
iii)
a) Number of times the account has been overdrawn since it was opened – Poisson Process / Counting Process
b) Status (overdrawn, in credit) of the account on the last day of each month – Markov Jump Chain/ Counting Process
c) number of direct debits paid since the account was opened – Poisson
d) Status (overdrawn, in credit) of the account at any time since the account was opened. Markov Jump Process

Q4)

i) Past history is needed to decide where to go in the chain.

If a sportsmen is at A and his/her performance reduces, you need to know what level of performance he was at the previous year to determine whether he or she drops one or two levels

ii) The B level needs to be split into two.
B+ is the level with no reduction in performance parameter last year
B- is the level with reduction in performance parameter last year
These levels are for modelling purposes and not the actual levels.



iv) The transition matrix is given by:

	D	C	B+	B-	A
D	0.5	0.5	0	0	0
C	0.3	0.2	0.5	0	0
B+	0	0.3	0.2	0	0.5
B-	0.3	0	0.2	0	0.5
A	0	0	0	0.3	0.7

v) Stationary distribution:
The stationary disctribution is the set of probabilities that satisfy the martrix equation
π = πP with and additional condition $\Sigma\pi\pi_{ii}$ = 1
Written out fully, this set of matrix equation is
$0.5\pi\pi_{DD}+0.3\pi\pi_{CC}+0.3\pi\pi_{BB-}=\pi\pi_{DD}$ ---a)
$0.5\pi\pi_{DD}+0.2\pi\pi_{CC}+0.3\pi\pi_{BB+}=\pi\pi_{CC}$ ---b)
$0.5\pi\pi_{CC}+0.2\pi\pi_{BB+}+0.2\pi\pi_{BB-}=\pi\pi_{BB+}$ ---c)
$0.3\pi\pi_{AA}=\pi\pi_{BB-}$ ---d)
$0.5\pi\pi_{BB+}+0.5\pi\pi_{BB-}+0.7\pi\pi_{AA}=\pi\pi_{AA}$ ---e)
Substituting (d) in (e)
$0.5\pi\pi_{BB+}+0.5*0.3\pi\pi_{AA}+0.7\pi\pi_{AA}=\pi\pi_{AA}$
=> $\pi\pi_{BB+}=0.3\pi\pi_{AA}$ …….(vi)
Substituing $\pi\pi_{BB+}\ aaaaaa\pi\pi_{AA}$ in (c) we get

$0.5\pi\pi_{CC}+0.2*0.3\pi\pi_{AA}+0.2*0.3\pi\pi_{AA}=0.3\pi\pi_{AA}$
=> $\pi\pi_{CC}=0.36\pi\pi_{AA}$
Substituing $\pi\pi_{BB+},\pi\pi_{CC}\ aaaaaa\pi\pi_{AA}$ in (b)
$0.5\pi\pi_{DD}+0.2*0.36\pi\pi_{AA}+0.3*0.3\pi\pi_{AA}=0.36\pi\pi_{AA}$ ---b)
=> $\pi\pi_{DD}=0.396\pi\pi_{AA}$
Substiting all in $\Sigma\pi\pi_{ii}$ = 1
$\pi\pi_{AA}+0.3\pi\pi_{AA}+0.3\pi\pi_{AA}+0.36\pi\pi_{AA}+0.396\pi\pi_{AA}=1$
- ⇨ $\pi\pi_{AA}$ = 0.42445
- ⇨ $\pi\pi_{BB-}$ = 0.12733
- ⇨ $\pi\pi_{BB+}$ = 0.12733
- ⇨ $\pi\pi_{CC}$ = 0.15280
- ⇨ $\pi\pi_{DD}$ = 0.1680 8

vi) Long run average contract value is
100% π_D + 120% π_C + 150% (π_{B+} + π_{B-}) + 175% π_A
= 1.4762 million USD

vii) Let m_i be the number of transitions (and years) taken to reach level A from any current level i. Then

mm_{DD}=1+0.5mm_{DD}+ 0.5mm_{CC} .....(a)
mm_{cc}=1+0.2mm_{CC}+ 0.5mm_{BB+}+ 0.3mm_{DD} .....(b)
mm_{BB+}=1+0.2mm_{BB+}+ 0.5mm_{AA}+ 0.3mm_{CC} .....(c)
mm_{BB-}=1+0.3mm_{DD}+ 0.5mm_{AA}+ 0.2mm_{BB+} .....(c)
mm_{AA} = 0
from (a) we get, mm_{DD}=2+mm_{CC}
substituting in (b) we get,
mm_{cc}=1+0.2mm_{CC}+ 0.5mm_{BB+}+ 0.3(2+mm_{CC})
mm_{BB+}= mm_{CC}−3.2
substituting in (c) ,
mm_{BB+}=1+0.2mm_{BB+}+ 0.5mm_{AA}+ 0.3mm_{CC}
mm_{CC}−3.2=1+0.2(mm_{CC}−3.2)+ 0.3mm_{CC}

mm_{CC} = 7.12 years
Therefore, mm_{DD} = 2+ mm_{CC} = 9.12 years
(note: this is an expected value and need not be rounded to integer number)

Q5

i) There is an explicit dependence on the past behavior of $\{Y_j: j\leq n\}$ in the probability distribution of Y_{n+1}; further, X_n is nothing but the sum of Y_j. Hence the Markov property does not hold.
ii) In part i), we show that there is explicit dependence on the past
behavior of $\{Y_j: j\leq n\}$ and hence the Markov property does not hold, this implies that that sequence $\{Y_n ; n \geq 1\}$ does not form a Markov chain.
iii) The transition matrix is given by:

$$\begin{vmatrix} p & 1-p & 0 & . & . & 0 \\ 0 & pe^{-\lambda} & 1-pe^{-\lambda} & 0 & . & . \\ . & 0 & pe^{-2\lambda} & 1-pe^{-2\lambda} & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & . & . \end{vmatrix}$$

iv)
(a) The chain is time homogeneous since the transition probabilities calculated in part i) is independent of time n.
(b) It is irreducible, since the number of errors can never go down.

(c) There are no recurrent states; hence there can be no stationary distribution.
Alternatively, if a stationary distribution π exists, it has to follow:

$\pi_0 p = \pi_0$

$\pi_0(1-p) + \pi_1 pe\hat{}(-\lambda) = \pi_1$

$\pi_1(1-pe\hat{}(-\lambda)) + \pi_2 pe\hat{}(-2\lambda) = \pi_2$

and so on.
Since $p \leq 1$, we have $\pi_0 = 0$ and then $\pi_1 = 0$, etc. Hence, no stationary distribution exists.

v) Probability of no further error is
$(pe\hat{}(-j\lambda))^n = p^n e\hat{}(-nj\lambda)$

Q7

(a) The process may be expressed as a Markov chain by considering following states. Each state indicates the current number of successive defeats.
State Number of successive defeats
1 0 – Not defeated last time
2 1 – Defeated in the last match
3 2 – Defeated in the last two matches
4 3 – Defeated in the last three matches
5 4 – Captain sacked
The transition matrix is as follows:

0.7	0.3	0	0	0
0.7	0	0.3	0	0
0.7	0	0	0.3	0
0.7	0	0	0	0.3
0	0	0	0	1

b) A Markov chain is said to be irreducible if any state *j* can be reached from any state *i*. The above process is not irreducible as the captain, once sacked, can never become the captain again.
(c)
(i) The probability of remaining the captain for exactly four matches is given by:

Match #	#1	#2	#3	#4	Probability
Result	Lose	Lose	Lose	Lose	
Probability	0.3	0.3	0.3	0.3	=0.3^4 = 0.0081

(ii) The probability of remaining the captain for exactly five matches is given by:

Match #	#1	#2	#3	#4	#5	Probability
Result	Not Lose	Lose	Lose	Lose	Lose	
Probability	0.7	0.3	0.3	0.3	0.3	0.00567

(iii) The probability of remaining the captain for exactly seven matches is given by:

Match #	#1	#2	#3	#4	#5	#6	#7	Probability
Result	Any	Any	Not Lose	Lose	Lose	Lose	Lose	
Probability	1	1	0.7	0.3	0.3	0.3	0.3	0.00567

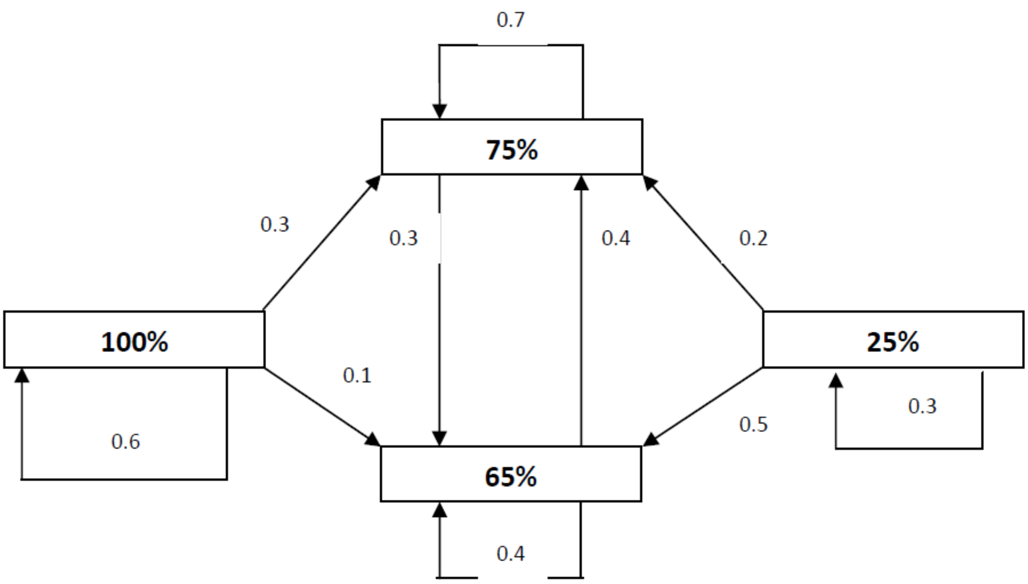
(iv) The probability of remaining the captain for exactly nine matches is given by:

Match #	#1 #2 #3 #4	#5	#6	#7	#8	#9	Probability
Result	Not all four losses	Not Lose	Lose	Lose	Lose	Lose	
Probability	=1-0.3^4=.9919	0.7	0.3	0.3	0.3	0.3	0.0056241

(d) Define N_i = Expected number of matches as a captain given that the current state is i. Since the captain is newly
appointed, the variable N in the question is N_1 as defined here.
We have:
$N_1 = 1 + 0.7 \times N_1 + 0.3 \times N_2$
$N_2 = 1 + 0.7 \times N_1 + 0.3 \times N_3$
$N_3 = 1 + 0.7 \times N_1 + 0.3 \times N_4$
$N_4 = 1 + 0.7 \times N_1$
Solving the above equations; we get:
$N_4 = 123.46$
$N_3 = 160.49$
$N_2 = 171.60$ and
$N_1 = 174.94$
Therefore, E(N) is 174.94 matches.

Q9

a) The state transition diagram is set out below:



b) We are to calculate for the % of corporate buyer having a target % for XYZ of 65%
- In 2 years time
- Over the long-run.

The state of the system after one year $S1 = S0P$

$$(0.05, 0.30, 0.45, 0.20)\begin{pmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{pmatrix} = (0.03, 0.445, 0.375, 0.15)$$

Hence the state of the system in 2 years time S2 = S1P

$$(0.03, 0.445, 0.375, 0.15)\begin{pmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{pmatrix} = (0.018, 0.5005, 0.3615, 0.12)$$

Hence the % of corporate buyer having a target % of 65% for XYZ in 2 years time is 36.15%.

The long run steady state can be found by solving the following equation:

S = SP

i.e.

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)P$$

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)\begin{pmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{pmatrix}$$

Hence we have the five equations

$$x_1 = 0.6x_1$$

$$x_2 = 0.3x_1 + 0.7x_2 + 0.4x_3 + 0.2x_4$$

$$x_3 = 0.1x_1 + 0.3x_2 + 0.4x_3 + 0.5x_4$$

$$x_4 = 0.2x_3 + 0.3x_4$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

Now from the first equation above,

$$0.4x_1 = 0$$

So

$$x_1 = 0$$

Substituting this into the other equations above and rearranging we get

$$0.3x_2 = 0.4x_3 + 0.2x_4 \quad (1)$$

$$0.6x_3 = 0.3x_2 + 0.5x_4 \quad (2)$$

$$0.7x_4 = 0.2x_3 \quad (3)$$

$$x_2 + x_3 + x_4 = 1 \quad (4)$$

One of the above equations is redundant.

Hence we have

$$(x_1 = 0, x_2 = 0.5423, x_3 = 0.3559, x_4 = 0.1017)$$

c) The steady state proportion of customers having a 65% target allocation for XYZ Ltd is 35.59%.
It is noted that the initially this proportion is 45% and it quickly drops to:

- 37.5% in one year's time; and
- 36.2% in two years' time

It is, therefore, likely that the steady state shall be reached in a few years' time.

Q11

i) If each room is represented by the state, then the transition matrix P for this Markov chain is as follows:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

ii) The chain is irreducible, because it is possible to go from any state to any other state. However, it is not aperiodic, because for any even n ,P6,1 n will be zero and for any odd n P6,5 n will also be zero . This means that there is no power of P that would have all its entries strictly positive.

iii) For P to be stationary, πP = P
Perform matrix multiplication and show that π P is equal to P

iv) We find from π that the mean recurrence time (i.e. the expected time to return) for the room 1 is 1/π(1)=12

v) Let, ψ(i) = E(number of steps to reach state 5 | X0 = i). We have ψ(5) = 0 ψ(6) = 1 + (1/2)ψ(5) + (1/2)ψ(4) ψ(4) = 1 + (1/2)ψ(6) + (1/2)ψ(3) ψ(3) = 1 + (1/4)ψ(1) + (1/4)ψ(2) + (1/4)ψ(4) + (1/4)ψ(5) ψ(1) = 1 + ψ(3) ψ(2) = 1 + ψ(3). [1.5] We solve and find ψ(1) = 7.